

# SAMPLING LOVÁSZ LOCAL LEMMA FOR GENERAL CONSTRAINT SATISFACTION SOLUTIONS IN NEAR-LINEAR TIME

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**ABSTRACT.** We give a fast algorithm for sampling uniform solutions of *general* constraint satisfaction problems (CSPs) in a local lemma regime. Suppose that the CSP has  $n$  variables with domain size at most  $q$ , each constraint contains at most  $k$  variables, shares variables with at most  $\Delta$  constraints, and is violated with probability at most  $p$  by a uniform random assignment. The algorithm returns an almost uniform satisfying assignment in expected  $\text{poly}(q, k, \Delta) \cdot \tilde{O}(n)$  time, as long as a local lemma condition is satisfied:

$$k \cdot p \cdot q^2 \cdot \Delta^5 \leq C_0 \quad \text{for a suitably small absolute constant } C_0.$$

Previously, under similar local lemma conditions, sampling algorithms with running time polynomial in both  $n$  and  $\Delta$  were only known for the almost atomic case, where each constraint is violated by a small number of forbidden local configurations. The key term  $\Delta^5$  in our local lemma condition also improves the previously best known  $\Delta^7$  for general CSPs [JPV21b] and  $\Delta^{5.714}$  for atomic CSPs, including the special case of  $k$ -CNF [JPV21a, HSW21].

Our sampling approach departs from previous fast algorithms for sampling LLL, which were based on Markov chains. A crucial step of our algorithm is a recursive marginal sampler that is of independent interests. Within a local lemma regime, this marginal sampler can draw a random value for a variable according to its marginal distribution, at a cost independent of the size of the CSP.

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## 1. INTRODUCTION

Constraint satisfaction problems (CSPs) are one of the most fundamental objects in computer science. A CSP is described by a collection of constraints defined on a set of variables. Formally, an instance of constraint satisfaction problem, called a *CSP formula*, is denoted by  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ . Here,  $V$  is a set of  $n = |V|$  variables;  $\mathcal{Q} \triangleq \bigotimes_{v \in V} Q_v$  is a product space of all assignments of variables, where each  $Q_v$  is a finite domain of size  $q_v \triangleq |Q_v| \geq 2$  over where the variable  $v$  ranges; and  $\mathcal{C}$  gives a collection of local constraints, such that each  $c \in \mathcal{C}$  is a constraint function  $c : \bigotimes_{v \in \text{vbl}(c)} Q_v \rightarrow \{\text{True}, \text{False}\}$  defined on a subset of variables, denoted by  $\text{vbl}(c) \subseteq V$ . An assignment  $\mathbf{x} \in \mathcal{Q}$  is called *satisfying* for  $\Phi$  if

$$\Phi(\mathbf{x}) \triangleq \bigwedge_{c \in \mathcal{C}} c(\mathbf{x}_{\text{vbl}(c)}) = \text{True}.$$

The followings are some key parameters of a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ :

- *domain size*  $q = q_\Phi \triangleq \max_{v \in V} |Q_v|$  and *width*  $k = k_\Phi \triangleq \max_{c \in \mathcal{C}} |\text{vbl}(c)|$ ;
- *constraint degree*  $\Delta = \Delta_\Phi \triangleq \max_{c \in \mathcal{C}} |\{c' \in \mathcal{C} \mid \text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset\}|$ ;<sup>1</sup>
- *violation probability*  $p = p_\Phi \triangleq \max_{c \in \mathcal{C}} \mathbb{P}[\neg c]$ , where  $\mathbb{P}$  denotes the law for the uniform assignment, in which each  $v \in V$  draws its evaluation from  $Q_v$  uniformly and independently at random.

The famous *Lovász Local Lemma (LLL)* [EL75] provides a sufficient criterion for the satisfiability of  $\Phi$ . Specifically, a satisfying assignment for a CSP formula  $\Phi$  exists if

$$(1) \quad ep\Delta \leq 1.$$

Due to a lower bound of Shearer [She85], such “LLL condition” for the existence of satisfying solution is essentially tight if only knowing  $p$  and  $\Delta$ . On the other hand, the *algorithmic* or *constructive LLL* seeks to find a solution efficiently. A major breakthrough was the Moser-Tardos algorithm [MT10], which guarantees to find a satisfying assignment efficiently under the LLL condition in (1).

**The sampling LLL.** We are concerned with the problem of *sampling Lovász Local Lemma*, which has drawn considerable attention in recent years [GJL19, Moi19, GLLZ19, GGGY20, Har20, FGYZ21, FHY21, JPV21a, JPV21b, HSW21, GGW22, FGW22]. In the context of CSP, it seeks to provide an efficient sampling algorithm for (nearly) uniform generation of satisfying assignments for the CSPs in an LLL-like regime. This sampling LLL problem is closely related to the problem of estimating the volume of solution spaces or the partition functions of statistical physics systems, and is motivated by fundamental tasks, including the probabilistic inferences in graphical models [Moi19] and the network reliability problems [GJL19, GJ19, GH20].

This problem of sampling LLL turns out to be computationally more challenging than the traditional algorithmic LLL, which requires constructing an arbitrary satisfactory assignment, not necessarily following the correct distribution. For example, when used as a sampling algorithm, the Moser-Tardos algorithm can only guarantee correct sampling on restrictive classes of CSPs [GJL19]. Due to the computational lower bounds shown in [BGG<sup>+</sup>19, GGW22], a strengthened LLL condition with  $c \geq 2$ :

$$(2) \quad p\Delta^c \lesssim 1,$$

is necessary for the tractability of sampling LLL, even restricted to typical specific sub-classes of CSPs, such as CNF or hypergraph coloring. Here  $\lesssim$  ignores the lower-order terms and the constant factor.

In a seminal work of Moitra [Moi19], a very innovative algorithm was given for sampling almost uniform  $k$ -CNF solutions assuming an LLL condition  $p\Delta^{60} \lesssim 1$ . This sampling algorithm was based on deterministic approximate counting by solving linear programs on properly factorized formulas and has a running time of  $n^{\text{poly}(k, \Delta)}$ . This LP-based approach was later extended to hypergraph coloring [GLLZ19] and random CNF formulas [GGGY20], and finally in a work of Jain, Pham and Vuong [JPV21b] to all CSPs satisfying a substantially improved LLL condition  $p\Delta^7 \lesssim 1$ . All these deterministic approximate counting based algorithms suffered from an  $n^{\text{poly}(k, \Delta)}$  time cost.

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<sup>1</sup>The constraint degree  $\Delta$  should be distinguished from the *dependency degree*  $D$ , which is the maximum degree of the dependency graph:  $D \triangleq \max_{c \in \mathcal{C}} |\{c' \in \mathcal{C} \setminus \{c\} \mid \text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset\}|$ . Note that  $\Delta = D + 1$ .

Historically, rapidly mixing Markov chains have been the canonical sampling algorithms, and often have near-linear time efficiency. However, for sampling LLL, there used to be a fundamental barrier for Markov chains. That is, despite the ubiquity of solutions, the solution space of CSPs may be highly disconnected through the transition of local Markov chains.

This barrier of disconnectivity was circumvented in a breakthrough of Feng *et al.* [FGYZ21], in which a rapidly mixing *projected* random walk was simulated efficiently on a subset of variables constructed using the marking/unmarking strategy invented in [Moi19]. Assuming an LLL condition  $p\Delta^{20} \lesssim 1$ , this new algorithm could generate an almost uniform  $k$ -CNF solution using a time cost within  $\text{poly}(k, \Delta) \cdot n^{1.0001}$ , which is close to linear in the number of variables  $n$ . By observing that this marking/unmarking of variables was, in fact a specialization in the Boolean case of compressing variables' states, this Markov chain based fast sampling approach was generalized in [FHY21] to CSPs beyond the Boolean domain, specifically, to all almost *atomic* CSPs (which we will explain later), assuming an LLL condition  $p\Delta^{350} \lesssim 1$ . This bound was remarkably improved to  $p\Delta^{7.04} \lesssim 1$  in another work of Jain, Pham and Vuong [JPV21a] through a very clever witness-tree-like information percolation analysis of the mixing time, which was also used later to support a perfect sampler through the coupling from the past (CFTP) in [HSW21] with a further improved condition  $p\Delta^{5.71} \lesssim 1$ .

All these fast algorithms for sampling LLL are restricted to the (almost) atomic CSPs, in which each constraint  $c$  is violated by exactly one (or very few) forbidden assignment(s) on  $\text{vbl}(c)$ .

**Challenges for general CSP.** New techniques are needed for fast sampling LLL for general CSPs. All existing fast algorithms for sampling LLL relied on some projection of the solution space to a much smaller space where the barrier of disconnectivity could be circumvented because the images of the projection might collide and were well connected. In order to efficiently simulate the random walk on the projected space and to recover a random solution from a random image, one would hope that the CSP formula were well “factorized” into small clusters most of the time because many constraints had already been satisfied for sure given the current image, which was indeed the case for fast sampling LLL for atomic CSPs [FGYZ21, FHY21, JPV21a, HSW21]. But for general non-atomic CSPs, it may no longer be the case, because now a bad event (violation of a constraint) may be highly non-elementary, and hence is no longer that easy to avoid cleanly after projection, which breaks the factorization.

It is possible that the non-atomicity of general CSPs might have imposed greater challenges to the sampling LLL than to its constructive counterpart. To see this, note that general CSPs can be simulated by atomic ones: by replacing each general constraint  $c$  having  $N$  forbidden assignments on  $\text{vbl}(c)$ , with  $N$  atomic constraints on the same  $\text{vbl}(c)$  each forbidding one assignment. Such simulation would increase the constraint degree  $\Delta$  by a factor of at most  $N$  and also decrease the violation probability  $p$  by a factor of  $N$ . For the classic LLL condition (1) where  $p$  and  $\Delta$  are homogeneous, this would not change the LLL condition; but the regime for the sampling LLL captured by (2) would be significantly reduced, since there  $p$  and  $\Delta$  are necessarily not homogeneous due to the lower bounds in [BGG<sup>+</sup>19, GGW22]. This situation seems to suggest that the non-atomicity of general CSPs might impose bigger challenges to the sampling LLL than to the existential/constructive LLL.

Indeed, prior to our work, it was not known for general CSPs with unbounded width  $k$  and degree  $\Delta$ , whether the sampling problem is polynomial-time tractable under an LLL condition like (2).

**1.1. Our results.** In this paper, we answer the above open question positively. We give a new algorithm that departs from all prior fast samplers based on Markov chains and achieves, for the first time, a fast sampling of almost uniform satisfying solutions for general CSPs in an improved local lemma regime.

As in the case of algorithmic LLL [MT10, HV15], we assume an abstraction of constraint evaluations, because arbitrary constraint functions defined on a super-constant number of variables can be highly nontrivial to express and evaluate. Specifically, we assume the following evaluation oracle for checking whether a constraint is already satisfied by a partially specified assignment.

**Assumption 1 (evaluation oracle).** There is an *evaluation oracle* for  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  such that given any constraint  $c \in \mathcal{C}$ , any assignment  $\sigma \in \mathcal{Q}_\Lambda \triangleq \bigotimes_{v \in \Lambda} Q_v$  specified on a subset  $\Lambda \subseteq \text{vbl}(c)$  of variables, the oracle answers whether  $c$  is already satisfied by  $\sigma$ , i.e.  $c(\tau) = \text{True}$  for all  $\tau \in \mathcal{Q}_{\text{vbl}(c)}$  that  $\tau_\Lambda = \sigma_\Lambda$ .

For specific classes of CSPs, e.g.  $k$ -CNF or hypergraph coloring, such an oracle is easy to realize.

Assuming such an oracle for constraint evaluations, we give the following fast, almost uniform sampler for general CSPs in a local lemma regime. Recall the parameters  $q, k, p, \Delta$  of a CSP formula  $\Phi$ .

**Theorem 1.1** (informal). *There is an algorithm such that given as input any  $\varepsilon \in (0, 1)$  and any CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  with  $n$  variables satisfying*

$$(3) \quad k \cdot p \cdot q^2 \cdot \Delta^5 \leq \frac{1}{150e^3},$$

*the algorithm terminates within  $\text{poly}(q, k, \Delta) \cdot n \log(\frac{n}{\varepsilon})$  time in expectation and outputs an almost uniform sample of satisfying assignments for  $\Phi$  within  $\varepsilon$  total variation distance.*

The formal statement of the theorem is in Theorem 5.1 (for termination and correctness of sampling) and in Theorem 6.3 (for efficiency of sampling).

The condition in (3) becomes  $p\Delta^{5+o(1)} \lesssim 1$  when  $p \leq (qk)^{-\omega(1)}$ , while a typical case is usually given by a much smaller  $p \leq q^{-\Omega(k)}$ . The previous best bound for sampling general CSP solutions was that  $q^3 k p \Delta^7 < c$  for a small constant  $c$ , achieved by the deterministic approximate counting based algorithm in [JPV21b] whose running time was  $(n/\varepsilon)^{\text{poly}(k, \Delta, \log q)}$ . We remark that our bound also improves the previous best bound,  $p\Delta^{5.714} \lesssim 1$ , for sampling almost atomic CSP and  $k$ -SAT [HSW21, JPV21a].

Let  $Z$  be the total number of satisfying assignments for  $\Phi$ . A  $\hat{Z}$  is called an  $\varepsilon$ -approximation of  $Z$  if  $(1 - \varepsilon)Z \leq \hat{Z} \leq (1 + \varepsilon)Z$ . By routinely going through the non-adaptive annealing process in [FGYZ21], the approximate sampler in Theorem 1.1 can be used as a black-box to give for any  $\varepsilon \in (0, 1)$  an  $\varepsilon$ -approximation of  $Z$  in time  $\text{poly}(q, k, \Delta) \cdot \tilde{O}(n^2 \varepsilon^{-2})$  with high probability.

1.1.1. *Perfect sampler.* The evaluation oracle in Assumption 1 in fact checks the sign of  $\mathbb{P}[\neg c \mid \sigma]$ , the probability that a constraint  $c$  is violated given a partially specified assignment  $\sigma$ . If further this probability can be estimated efficiently, then the sampling in Theorem 1.1 can be made perfect, where the output sample follows exactly the target distribution.

**Theorem 1.2** (informal). *For the input class of CSPs, if there is such an FPTAS for violation probability:*

- *for any constraint  $c \in \mathcal{C}$ , any assignment  $\sigma \in \mathcal{Q}_\Lambda$  specified on a subset  $\Lambda \subseteq \text{vbl}(c)$ , and  $0 < \varepsilon < 1$ , an  $\varepsilon$ -approximation of  $\mathbb{P}[\neg c \mid \sigma]$  is returned deterministically within  $\text{poly}(q, k, 1/\varepsilon)$  time,*

*then the sampling algorithm in Theorem 1.1 returns a perfect sample of uniform satisfying assignment within  $\text{poly}(q, k, \Delta) \cdot n$  time in expectation under the same condition (3).*

The formal statement of the theorem is in Theorem 5.1 (for termination and correctness of sampling) and in Theorem 6.1 (for efficiency of sampling). In fact, we prove this perfect sampler first, and then realize the FPTAS assumed in Theorem 1.2 using Monte Carlo experiments, which introduces a bounded bias to the sampling and gives us the approximate sampler claimed in Theorem 1.1.

For concrete classes of CSPs defined by simple local constraints, it is no surprise to see that the probability  $\mathbb{P}[\neg c \mid \sigma]$  almost always has an easy-to-compute closed-form expression, in which case we have a perfect sampler without assuming the oracles in Assumption 1 and in Theorem 1.2.

The followings are two examples of non-atomic CSPs which admit linear-time perfect samplers.

**Example 1.3** ( $\delta$ -robust  $k$ -SAT). *The  $n$  variables are Boolean, each clause contains exactly  $k$  literals, and a clause is satisfied if and only if at least  $\delta k$  of its literals have the outcome True.*

- *For  $\delta$ -robust  $k$ -SAT with variable degree  $d$  (each variable appears in at most  $d$  clauses) satisfying*

$$0 < \delta < \frac{1}{2}, \quad k \geq \frac{24 \ln k + 20 \ln d + 40}{(1 - 2\delta)^2},$$

*a perfect sample of uniform satisfying solutions is returned within expected time  $\text{poly}(k, d) \cdot n$ .*

**Example 1.4** ( $\delta$ -robust hypergraphs  $q$ -coloring). *Each vertex is colored with one of the  $q$  colors, each hyperedge is  $k$ -uniform and is satisfied if and only if there are no  $(1 - \delta)k$  vertices with the same color.*

- For  $k$ -uniform hypergraphs on  $n$  vertices with maximum vertex degree  $d$  satisfying

$$(1 - \delta)k \geq 15, \quad q \geq \frac{7d^{\frac{5}{(1-\delta)k-3}} \cdot 4^{\frac{1}{(1-\delta)}}}{(1 - \delta)^{1.25}},$$

a perfect sample of uniform satisfying coloring is returned within expected time  $\text{poly}(q, k, d) \cdot n$ .

1.1.2. *Marginal sampler.* The core component of our sampling algorithm is a *marginal sampler* for drawing from marginal distributions. Let  $\mu = \mu_\Phi$  denote the uniform distribution over all satisfying assignments for  $\Phi$ , and for each  $v \in V$ , let  $\mu_v$  denote the marginal distribution at  $v$  induced by  $\mu$ .

**Theorem 1.5** (informal). *There is an algorithm such that given as input any  $\varepsilon \in (0, 1)$ , any CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  satisfying (3), and any  $v \in V$ , the algorithm returns a random value  $x \in Q_v$  distributed approximately as  $\mu_v$  within total variation distance  $\varepsilon$ , within  $\text{poly}(q, k, \Delta, \log(1/\varepsilon))$  time in expectation.*

This marginal sampler is also perfect under the same assumption as in Theorem 1.2. Another byproduct of this marginal sampler is the following algorithm for probabilistic inference.

**Theorem 1.6** (informal). *There is an algorithm such that given as input any  $\varepsilon, \delta \in (0, 1)$ , any CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  satisfying (3), and any  $v \in V$ , the algorithm returns for every  $x \in Q_v$  an  $\varepsilon$ -approximation of the marginal probability  $\mu_v(x)$  within  $\text{poly}(q, k, \Delta, 1/\varepsilon, \log(1/\delta))$  time with probability at least  $1 - \delta$ .*

The above two theorems are formally restated and proved in Theorem 6.22.

By a self-reduction, the sampling and inference algorithms in Theorems 1.5 and 1.6 remain to hold for the marginal distributions  $\mu_v^\sigma$  conditioning on a feasible partially specified assignment  $\sigma$ , as long as the LLL condition (3) is satisfied by the new instance  $\Phi^\sigma$  obtained from pinning  $\sigma$  onto  $\Phi$ .

Both the above algorithms for marginal sampling and probabilistic inference are **local algorithms whose costs are independent of  $n$** . Previously, in order to simulate or estimate the marginal distribution of a variable, it was often necessary to generate a full assignment on all  $n$  variables, or at least pay no less than that. One might have asked the following natural question:

*Can these locally defined sampling or inference problems be solved at a local cost?*

However, decades have passed, and only recently has such a novel local algorithm been discovered for marginal distributions in infinite spin systems [AJ22], which is also our main source of inspiration.

1.2. **Technique overview.** As we have explained before, non-atomicity of constraints causes a barrier for the current Markov chain based algorithms [FGYZ21, FHY21, JPV21a, HSW21, FGW22]. There is another family of sampling algorithms, which we call “resampling based” algorithms [FH00, GJL19, FVY19, Jer21, FGY22]. These algorithms use resampling of variables to fix the assignment until it follows the right distribution, morally like the Moser-Tardos algorithm, and they are not as affected by disconnectivity of solution space as Markov chains, but here a principle to ensure the correct sampling is to resample the variables that the algorithm has observed and conditioned on, which also causes trouble on non-atomic constraints, because to ensure such constraints are satisfied, the algorithm has to observe too many variables, whose resampling would cancel the progress of the algorithm.

We adopt a new idea of sampling, which we call the *recursive marginal sampler*. It is somehow closer to the resampling based algorithms than to the Markov chains, but thanks to its recursive nature, the algorithm avoids excessive resampling. This algorithm is inspired by a recent novel algorithm of Anand and Jerrum [AJ22] for perfectly sampling in infinite spin systems, where a core component is such a marginal sampler that can draw a spin according to its marginal distribution.

Now let us consider the uniform distribution  $\mu$  over all satisfying assignments of a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ , and its marginal distribution  $\mu_v$  at a variable  $v \in V$ , say over domain  $Q_v = [q]$ . To sample from this  $\mu_v$  over  $[q]$ , an idea is to exploit the so-called “local uniformity” property [HSS11], which basically says that  $\mu_v$  should not be far from a uniform distribution over  $[q]$  in total variation distance when  $\Phi$  satisfies some local lemma condition. Therefore, a uniform sample from  $[q]$  already gives a coarse sample of  $\mu_v$ . It remains to boost such a coarse sampler to a sampler with arbitrary precision.

By the local uniformity, there exists a  $\theta < \frac{1}{q}$  close enough to  $\frac{1}{q}$ , such that

$$(4) \quad \forall x \in [q], \quad \mu_v(x) \geq \theta.$$

The marginal distribution  $\mu_v$  can then be divided as

$$q\theta \cdot \mathcal{U} + (1 - q\theta) \cdot \mathcal{D},$$

where  $\mathcal{U}$  is the uniform distribution over  $[q]$  and  $\mathcal{D}$  gives a distribution of “overflow” mass such that:

$$\forall x \in [q], \quad \mathcal{D}(x) = \frac{\mu_v(x) - \theta}{1 - q\theta}.$$

Sampling from  $\mu_v$  then can follow this strategy: with probability  $q\theta$ , the algorithm falls into the “zone of local uniformity” and returns a uniform sample from  $\mathcal{U}$ ; and with probability  $1 - q\theta$ , the algorithm falls into the “zone of indecision” and has to draw a sample from this overflow distribution  $\mathcal{D}$ , which can be done by constructing a Bernoulli factory that accesses  $\mu_v$  as an oracle. But wait, if we had such an oracle for  $\mu_v$  in the first place, why would sampling from  $\mu_v$  even be a problem?

The above “chicken or egg” paradox is somehow resolved by a simple observation: if enough many other variables had already been sampled correctly, say with outcome  $X$ , then assuming a strong enough LLL condition, there is a good chance that the resulting formula  $\Phi^X$  was “factorized” into small clusters, from where a standard rejection sampling on  $\Phi^X$  would be efficient for sampling from  $\mu_v^X$ , and overall from  $\mu_v$ . Therefore, the sampling strategy is now corrected as: after falling into the “zone of indecision” and before trying to draw from the overflow distribution  $\mathcal{D}$ , the algorithm picks another variable  $u$  whose successful sampling might help factorize  $\Phi$ , and recursively apply the marginal sampler at  $u$  to draw from  $u$ ’s current marginal distribution first. The only loose end now is that the LLL condition is not self-reducible, meaning it is not invariant under arbitrary pinning. We adopt the idea of “freezing” constraints used in [JPV21b] to guide the algorithm to pick variables for sampling. The LLL condition is replaced by a more refined invariant condition that guarantees for each variable picked for sampling, the same local uniformity as in (4) to persist throughout the algorithm, and also guarantees a good chance of factorization while there are no other variables to pick.

To show the fast convergence of the recursive sampler, in [AJ22] the strategy was to show that the branching process given by the recursion tree always has decaying offspring number in expectation given the worst-case boundary condition, which is not true here. Instead, we apply a more average-case style analysis and bound the expected cost of the sampler according to the recursion tree directly.

To achieve a sharper bound, we design a new combinatorial structure named generalized  $\{2, 3\}$ -tree. In most works on counting/sampling LLL, two types of bad events are considered: one is that the assignment of a marked variable does not fall into the zone of local uniformity; the other is that a constraint is still not satisfied after that a large proportion of its variables are assigned [Moi19, FHY21, GLLZ19, JPV21b]. In previous work, these two bad events are treated similarly and bounded using a combinatorial structure named  $\{2, 3\}$ -tree [Alo91]. A crucial observation is that the densities of these two types of bad events are different, which inspires our design of this new combinatorial structure to take advantage of this property and push the bounds beyond state-of-the-arts.

## 2. NOTATIONS FOR CSP

We recall the definition of CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  in Section 1. We use  $\Omega = \Omega_\Phi$  to denote the set of all satisfying assignments of  $\Phi$ , and use  $\mu = \mu_\Phi$  to denote the uniform distribution over  $\Omega$ . Recall that  $\mathbb{P}$  denotes the law for the uniform product distribution over  $\mathcal{Q}$ . For  $C \subseteq \mathcal{C}$ , denote  $\text{vbl}(C) \triangleq \bigcup_{c \in C} \text{vbl}(c)$ ; and for  $\Lambda \subseteq V$ , denote  $\mathcal{Q}_\Lambda \triangleq \bigotimes_{v \in \Lambda} \mathcal{Q}_v$ . We introduce a notation for partial assignments.

**Definition 2.1** (partial assignment). Given a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ , define:

$$\mathcal{Q}^* \triangleq \bigotimes_{v \in V} (\mathcal{Q}_v \cup \{\star, \star\}),$$

where  $\star$  and  $\star$  are two special symbols not in any  $\mathcal{Q}_v$ . Each  $\sigma \in \mathcal{Q}^*$  is called a *partial assignment*.

In a partial assignment  $\sigma \in \mathcal{Q}^*$ , each variable  $v \in V$  is classified as follows:

- $\sigma(v) \in \mathcal{Q}_v$  means that  $v$  is *accessed* by the algorithm and *assigned* with the value  $\sigma(v) \in \mathcal{Q}_v$ ;
- $\sigma(v) = \star$  means that  $v$  is just *accessed* by the algorithm but *unassigned* yet with a value in  $\mathcal{Q}_v$ ;
- $\sigma(v) = \star$  means that  $v$  is *unaccessed* by the algorithm and hence *unassigned* with any value.

Furthermore, we use  $\Lambda(\sigma) \subseteq V$  and  $\Lambda^+(\sigma) \subseteq V$  to respectively denote the sets of assigned and accessed variables in a partial assignment  $\sigma \in \mathcal{Q}^*$ , that is:

$$(5) \quad \Lambda(\sigma) \triangleq \{v \in V \mid \sigma(v) \in \mathcal{Q}_v\} \quad \text{and} \quad \Lambda^+(\sigma) \triangleq \{v \in V \mid \sigma(v) \neq \star\}.$$

Given any partial assignment  $\sigma \in \mathcal{Q}^*$  and variable  $v \in V$ , we further denote by  $\sigma_{v \leftarrow x}$  the partial assignment obtained from modifying  $\sigma$  by replacing  $\sigma(v)$  with  $x \in \mathcal{Q}_v \cup \{\star, \star\}$ .

A partial assignment  $\tau \in \mathcal{Q}^*$  is said to *extend* a partial assignment  $\sigma \in \mathcal{Q}^*$  if  $\Lambda(\sigma) \subseteq \Lambda(\tau)$ ,  $\Lambda^+(\sigma) \subseteq \Lambda^+(\tau)$ , and  $\sigma, \tau$  agree with each other over all variables in  $\Lambda(\sigma)$ . A partial assignment  $\sigma \in \mathcal{Q}^*$  is said to satisfy a constraint  $c \in \mathcal{C}$  if  $c$  is satisfied by all full assignments  $\tau \in \mathcal{Q}$  that extend  $\sigma$ . A partial assignment  $\sigma \in \mathcal{Q}^*$  is called *feasible* if there is a satisfying assignment  $\tau \in \Omega$  that extends  $\sigma$ .

Given any feasible  $\sigma \in \mathcal{Q}^*$  and any  $S \subseteq V$ , we use  $\sigma_S$  to denote  $\bigotimes_{v \in S} \sigma(v)$  and  $\mu_S^\sigma$  to denote the marginal distribution induced by  $\mu$  on  $S$  conditioning on  $\sigma$ . For each  $\tau \in \mathcal{Q}_S$ , we have  $\mu_S^\sigma(\tau) = \Pr_{X \sim \mu} [X_S = \tau \mid \forall v \in \Lambda(\sigma), X(v) = \sigma(v)]$ . We further write  $\mu_v^\sigma = \mu_{\{v\}}^\sigma$ . Similar notation is used for the law  $\mathbb{P}$  for the uniform product distribution over  $\mathcal{Q}$ . For  $\sigma \in \mathcal{Q}^*$  and any event  $A \subseteq \mathcal{Q}$ , we have  $\mathbb{P}[A \mid \sigma] = \Pr_{X \in \mathcal{Q}} [X \in A \mid \forall v \in \Lambda(\sigma), X(v) = \sigma(v)]$ .

### 3. THE SAMPLING ALGORITHM

We give our main algorithm for sampling almost uniform satisfying assignments for a CSP formula. Our presentation uses notations defined in Section 2.

**3.1. The main sampling algorithm.** Our main sampling algorithm takes as input a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  with domain size  $q = q_\Phi$ , width  $k = k_\Phi$ , constraint degree  $\Delta = \Delta_\Phi$ , and violation probability  $p = p_\Phi$ , where the meaning of these parameters are as defined in Section 1.

We suppose that the  $n = |V|$  variables are enumerated as  $V = \{v_1, v_2, \dots, v_n\}$  in an arbitrary order. The CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  is presented to the algorithm by the evaluation oracle in Assumption 1. Also assume that given any  $c \in \mathcal{C}$  (or  $v \in V$ ), the  $\text{vbl}(c)$  (or  $\{c \in \mathcal{C} \mid v \in \text{vbl}(c)\}$ ) can be retrieved.

The main algorithm (Algorithm 1) is the same as the main sampling frameworks in [JPV21b, GLLZ19]. A partial assignment  $X \in \mathcal{Q}^*$  is maintained, initially as the empty assignment  $X = \star^V$ .

- (1) In the 1st phase, at each step it adaptively picks (in a predetermined order) a variable  $v$  that has enough “freedom” because it is not involved in any easy-to-violate constraint given the current  $X$ , and replaces  $X(v)$  with a random value drawn by a subroutine `MarginSample` according to the correct marginal distribution  $\mu_v^X$ .
- (2) When no such variable with enough freedom remains, the formula is supposed to be “factorized” enough into small clusters and the algorithm enters the 2nd phase, from where the partial assignment constructed in the 1st phase is completed to a uniform random satisfying assignment by a standard `RejectionSampling` subroutine.

A key threshold  $\alpha$  for the violation probability is fixed as below:

$$(6) \quad \alpha = \left(18e^2 q^2 k \Delta^2\right)^{-1},$$

which satisfies  $\alpha > p = p_\Phi$ , assuming the LLL condition in (3).

For the ease of exposition, we assume an oracle for approximately deciding whether a constraint becomes too easy to violate given the current partial assignment.

**Assumption 2.** There is an oracle such that given any partial assignment  $\sigma \in \mathcal{Q}^*$  and any constraint  $c \in \mathcal{C}$ , the oracle distinguishes between the two cases:  $\mathbb{P}[\neg c \mid \sigma] > \alpha$  and  $\mathbb{P}[\neg c \mid \sigma] < 0.99\alpha$ , and answers arbitrarily and consistently if otherwise, which means that the answer to the undefined case  $\mathbb{P}[\neg c \mid \sigma] \in [0.99\alpha, \alpha]$  can be either “yes” or “no” but remains the same for the same  $\sigma_{\text{vbl}(c)}$ .

Such an oracle is clearly implied by the FPTAS for violation probability assumed in Theorem 1.2 and will be explicitly realized later in Section 6. For now, with respect to such an oracle, the classes of easy-to-violate constraints and their involved variables are defined as follows.

**Definition 3.1** (frozen and fixed). Assume Assumption 2. Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment.

- A constraint  $c \in \mathcal{C}$  is called  $\sigma$ -frozen if it is reported  $\mathbb{P}[-c \mid \sigma] > \alpha$  by the oracle in Assumption 2. Denote by  $\mathcal{C}_{\text{frozen}}^\sigma$  the set of all  $\sigma$ -frozen constraints:

$$\mathcal{C}_{\text{frozen}}^\sigma \triangleq \{c \in \mathcal{C} \mid c \text{ is reported by the oracle to satisfy } \mathbb{P}[-c \mid \sigma] > \alpha\}.$$

- A variable  $v \in V$  is called  $\sigma$ -fixed if  $v$  is accessed in  $\sigma$  or is involved in some  $\sigma$ -frozen constraint. Denote by  $V_{\text{fix}}^\sigma$  the set of all  $\sigma$ -fixed variables:

$$V_{\text{fix}}^\sigma \triangleq \Lambda^+(\sigma) \cup \bigcup_{c \in \mathcal{C}_{\text{frozen}}^\sigma} \text{vbl}(c).$$

Similar ideas of freezing appeared in previous works on sampling and algorithmic LLL [JPV21b, Bec91].

**Remark 3.2** (one-sided error for frozen/fixed decision). By the property of the oracle in Assumption 2, any constraint  $c \in \mathcal{C}$  with  $\mathbb{P}[-c \mid \sigma] > \alpha$  must be in  $\mathcal{C}_{\text{frozen}}^\sigma$ , and any variable  $v \in V$  involved in such a constraint must be in  $V_{\text{fix}}^\sigma$ ; conversely, any  $\sigma$ -frozen constraint  $c \in \mathcal{C}_{\text{frozen}}^\sigma$  must have  $\mathbb{P}[-c \mid \sigma] \geq 0.99\alpha$  and any unaccessed  $\sigma$ -fixed variable  $v \in V_{\text{fix}}^\sigma$  must be involved in at least one of such constraints.

---

**Algorithm 1:** The sampling algorithm

---

**Input:** a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ ;

**Output:** a uniform random satisfying assignment  $X \in \Omega_\Phi$ ;

- 1  $X \leftarrow \star^V$ ;
  - 2 **for**  $i = 1$  to  $n$  **do**
  - 3     **if**  $v_i$  is not  $X$ -fixed **then**
  - 4          $X(v_i) \leftarrow \text{MarginSample}(\Phi, X, v_i)$ ;
  - 5  $X_{V \setminus \Lambda(X)} \leftarrow \text{RejectionSampling}(\Phi, X, V \setminus \Lambda(X))$ ;
  - 6 **return**  $X$ ;
- 

The following invariant is satisfied in the **for** loop in Algorithm 1 (formally proved in Lemma 5.4). The correctness of the MarginSample subroutine is guaranteed by this invariant.

**Condition 3.3** (invariant for MarginSample). *The following holds for the input tuple  $(\Phi, \sigma, v)$ :*

- $\Phi = (V, \mathcal{Q}, \mathcal{C})$  is a CSP formula,  $\sigma \in \mathcal{Q}^*$  is a feasible partial assignment, and  $v \in V$  is a variable;
- $v$  is not  $\sigma$ -fixed and  $\sigma(v) = \star$ , and for all  $u \in V$ ,  $\sigma(u) \in \mathcal{Q}_u \cup \{\star\}$ ;
- $\mathbb{P}[-c \mid \sigma] \leq \alpha q$  for all  $c \in \mathcal{C}$ .

The correctness of Algorithm 1 follows from the correctness of MarginSample and RejectionSampling for sampling from the correct marginal distributions, which is formally proved in Theorem 5.1.

In fact, the sampling in Algorithm 1 is *perfect*. It will only become approximate after the oracle in Assumption 2 realized by a Monte Carlo program that may bias the sampling.

**3.2. The rejection sampling.** We first introduce the RejectionSampling, which is a standard procedure. Our rejection sampling takes advantages of simplification and decomposition of a CSP formula.

A simplification of  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  under partial assignment  $\sigma \in \mathcal{Q}^*$ , denoted by  $\Phi^\sigma = (V^\sigma, \mathcal{Q}^\sigma, \mathcal{C}^\sigma)$ , is a new CSP formula such that  $V^\sigma = V \setminus \Lambda(\sigma)$  and  $\mathcal{Q}^\sigma = \mathcal{Q}_{V \setminus \Lambda(\sigma)}$ , and the  $\mathcal{C}^\sigma$  is obtained from  $\mathcal{C}$  by:

- (1) removing all the constraints that have already been satisfied by  $\sigma$ ;
- (2) for the remaining constraints, replacing the variables  $v \in \Lambda(\sigma)$  with their values  $\sigma(v)$ .

It is easy to see that  $\mu_{\Phi^\sigma} = \mu_{V \setminus \Lambda(\sigma)}^\sigma$  for the uniform distribution  $\mu_{\Phi^\sigma}$  over satisfying assignments of  $\Phi^\sigma$ .

A CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  can be naturally represented as a (multi-)hypergraph  $H_\Phi$ , where each variable  $v \in V$  corresponds to a vertex in  $H_\Phi$  and each constraint  $c \in \mathcal{C}$  corresponds to a hyperedge  $\text{vbl}(c)$  in  $H_\Phi$ . We slightly abuse the notation and write  $H_\Phi = (V, \mathcal{C})$ .

Let  $H_i = (V_i, \mathcal{C}_i)$  for  $1 \leq i \leq K$  denote all  $K \geq 1$  connected components in  $H_\Phi$ , and  $\Phi_i = (V_i, \mathcal{Q}_{V_i}, \mathcal{C}_i)$  their formulas. Obviously  $\Phi = \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_K$  with disjoint  $\Phi_i$ , and  $\mu_\Phi$  is the product of all  $\mu_{\Phi_i}$ . Also  $\mu_S$  on a subset of variables  $S \subseteq V$  is determined only by those components with  $V_i$  intersecting  $S$ .



---

**Algorithm 2:** RejectionSampling( $\Phi, \sigma, S$ )

---

**Input:** a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ , a feasible partial assignment  $\sigma \in \mathcal{Q}^*$ , and a subset  $S \subseteq V \setminus \Lambda(\sigma)$  of unassigned variables in  $\sigma$ ;  
**Output:** an assignment  $X_S \in \mathcal{Q}_S$  distributed as  $\mu_S^\sigma$ ;  
1 find all the connected components  $\{H_i^\sigma = (V_i^\sigma, \mathcal{C}_i^\sigma) \mid 1 \leq i \leq K\}$  in  $H_{\Phi^\sigma}$  s.t.  $V_i^\sigma$  intersects  $S$ , where  $\Phi^\sigma$  denotes the simplification of  $\Phi$  under  $\sigma$ ;  
2 **for**  $1 \leq i \leq K$  **do**  
3     **repeat**  
4         generate  $X_{V_i^\sigma} \in \mathcal{Q}_{V_i^\sigma}$  uniformly and independently at random;  
5     **until** all the constraints in  $\mathcal{C}_i^\sigma$  are satisfied by  $X_{V_i^\sigma}$ ;  
6 **return**  $X_S$  where  $X$  is the concatenation of all  $X_{V_i^\sigma}$ ;

---

For each  $\sigma \in \mathcal{Q}^*$  and  $v \in V^\sigma$ , let  $H_v^\sigma = (V_v^\sigma, \mathcal{C}_v^\sigma)$  denote the connected component in  $H^\sigma$  that contains the vertex/variable  $v$ . This definition will be useful later.

Our rejection sampling algorithm for drawing from a marginal distribution  $\mu_S^\sigma$  is given in Algorithm 2. The correctness of this algorithm is folklore. We state without proof.

**Theorem 3.4.** *On any input  $(\Phi, \sigma, S)$  as specified in Algorithm 2, RejectionSampling terminates with probability 1, and upon termination it returns an assignment  $X_S \in \mathcal{Q}_S$  distributed as  $\mu_S^\sigma$ .*

**3.3. The marginal sampler.** We now introduce the the core part of our sampling algorithm, the MarginSample subroutine. This procedure is a “marginal sampler”: it can draw a random value for a variable  $v \in V$  according to its marginal distribution  $\mu_v^\sigma$ . Our marginal sampling algorithm is inspired by a recent novel sampling algorithm of Anand and Jerrum for infinite spin systems [AJ22].

For each variable  $v \in V$ , we suppose that an arbitrary order is assumed over all values in  $Q_v$ ; we use  $q_v \triangleq |Q_v|$  to denote the domain size of  $v$  and fix the following parameters:

$$(7) \quad \theta_v \triangleq \frac{1}{q_v} - \eta - \zeta \quad \text{and} \quad \theta \triangleq \frac{1}{q} - \eta - \zeta \quad \text{where} \quad \begin{cases} \eta = (1 - \epsilon \alpha q)^{-\Delta} - 1 \\ \zeta = (16\epsilon q k \Delta)^{-1} \end{cases}$$

Note that  $\zeta < \frac{1}{q} - \eta$  is guaranteed by the LLL condition in (3), and hence  $\theta_v, \theta > 0$ .

The MarginSample subroutine for drawing from a marginal distribution  $\mu_v^\sigma$  is given in Algorithm 3.

---

**Algorithm 3:** MarginSample( $\Phi, \sigma, v$ )

---

**Input:** a CSP formula  $\Phi = (V, \mathcal{C})$ , a feasible partial assignment  $\sigma \in \mathcal{Q}^*$ , and a variable  $v \in V$ ;  
**Output:** a random  $x \in Q_v$  distributed as  $\mu_v^\sigma$ ;  
1 choose  $r \in [0, 1)$  uniformly at random;  
2 **if**  $r < q_v \cdot \theta_v$  **then** //  $r$  falls into the zone of local uniformity  
3     **return** the  $\lceil r/\theta_v \rceil$ -th value in  $Q_v$ ;  
4 **else** //  $r$  falls into the zone of indecision  
5     **return** MarginOverflow( $\Phi, \sigma_{v \leftarrow \star}, v$ );

---

An invariant satisfied by Algorithm 3 guarantees that  $\theta_v$  always lower bounds the marginal probability with gap  $\zeta$ . This is formally proved in Section 4 by a “local uniformity” property (Corollary 4.3).

**Proposition 3.5.** *Assuming Condition 3.3 for the input  $(\Phi, \sigma, v)$ , it holds that  $\min_{x \in Q_v} \mu_v^\sigma(x) \geq \theta_v + \zeta$ .*

Therefore, the function  $\mathcal{D}$  constructed below is a well-defined distribution over  $Q_v$ :

$$(8) \quad \forall x \in Q_v, \quad \mathcal{D}(x) \triangleq \frac{\mu_v^\sigma(x) - \theta_v}{1 - q_v \cdot \theta_v}.$$

Consider the following thought experiment. Partition  $[0, 1)$  into  $(q_v + 1)$  intervals  $I_1, I_2, \dots, I_{q_v}$  and  $I'$ , where  $I_i = [(i - 1)\theta_v, i\theta_v)$  for  $1 \leq i \leq q_v$  are of equal size  $\theta_v$ , and  $I' = [q_v \cdot \theta_v, 1)$  is the remaining

part. We call  $\bigcup_{i=1}^{q_v} I_i = [0, q_v \cdot \theta_v)$  the “zone of local uniformity” and  $I' = [q_v \cdot \theta_v, 1)$  the “zone of indecision”.

Drawing from  $\mu_v^\sigma$  can then be simulated as: first drawing a uniform random  $r \in [0, 1)$ , if  $r < q_v \cdot \theta_v$ , i.e. it falls into the “zone of local uniformity”, then returning the  $i$ -th value in  $Q_v$  if  $r \in I_i$ ; if otherwise  $r \in I'$ , i.e. it falls into the “zone of indecision”, then returning a random value drawn from the above  $\mathcal{D}$ . It is easy to verify that the generated sample is distributed as  $\mu_v^\sigma$ . And this is exactly what Algorithm 3 is doing, assuming that the subroutine  $\text{MarginOverflow}(\Phi, \sigma_{v \leftarrow \star}, v)$  correctly draws from  $\mathcal{D}$ .

**3.4. Recursive sampling for margin overflow.** The goal of the  $\text{MarginOverflow}$  subroutine is to draw from the distribution  $\mathcal{D}$  which is computed from the marginal distribution  $\mu_v^\sigma$  as defined in (8). Now suppose that we are given access to an oracle for drawing from  $\mu_v^\sigma$  (such an oracle can be realized by  $\text{RejectionSampling}(\Phi, \sigma, \{v\})$  in Algorithm 2). Then, drawing from  $\mathcal{D}$  that is a linear function of  $\mu_v^\sigma$ , by accessing an oracle for drawing from  $\mu_v^\sigma$ , can be resolved using the existing approaches of *Bernoulli factory* [NP05, Hub16, DHKN17].

This sounds silly because if such oracle for  $\mu_v^\sigma$  were efficient we would have been using it to output a sample for  $\mu_v^\sigma$  in the first place, which is exactly the reason why we ended up trying to draw from  $\mathcal{D}$ .

Nevertheless, such Bernoulli factory for sampling from  $\mathcal{D}$  may serve as the basis of a recursion, where sufficiently many variables with enough “freedom” would have been sampled successfully in their zones of local uniformity during the recursion, and hence the remaining CSP formula would have been “factorized” into small connected components, in which case an oracle for  $\mu_v^\sigma$  would be efficient to realize by the  $\text{RejectionSampling}(\Phi, \sigma, \{v\})$ , and the Bernoulli factory for  $\mathcal{D}$  could apply.

We define a class of variables that are candidates for sampling with priority in the recursion.

**Definition 3.6** ( $\star$ -influenced variables). Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment. Let  $H^\sigma = H_{\Phi\sigma} = (V^\sigma, \mathcal{C}^\sigma)$  be the hypergraph for simplification  $\Phi^\sigma$ . Let  $H_{\text{fix}}^\sigma$  be the sub-hypergraph of  $H^\sigma$  induced by  $V^\sigma \cap V_{\text{fix}}^\sigma$ .

- Let  $V_{\star\text{-con}}^\sigma \subseteq V^\sigma \cap V_{\text{fix}}^\sigma$  be the set of vertices belong to the connected components in  $H_{\text{fix}}^\sigma$  that contain any  $v$  with  $\sigma(v) = \star$ .
- Let  $V_{\star\text{-inf}}^\sigma \triangleq \{u \in V^\sigma \setminus V_{\star\text{-con}}^\sigma \mid \exists c \in \mathcal{C}^\sigma, v \in V_{\star}^\sigma : u, v \in \text{vbl}(c)\}$  be the vertex boundary of  $V_{\star\text{-con}}^\sigma$  in  $H^\sigma$ .
- Let  $\mathcal{C}_{\star\text{-con}}^\sigma$  be the set of constraints  $c \in \mathcal{C}$  that intersect  $V_{\star\text{-con}}^\sigma$ .
- Define  $\text{NextVar}(\sigma)$  by

$$(9) \quad \text{NextVar}(\sigma) \triangleq \begin{cases} v_i \in V_{\star\text{-inf}}^\sigma \text{ with smallest } i & \text{if } V_{\star\text{-inf}}^\sigma \neq \emptyset, \\ \perp & \text{otherwise.} \end{cases}$$

**Remark 3.7.** The  $\mathcal{C}_{\star\text{-con}}^\sigma$  defined above is not used here, but is important in the analysis. Same as in Definition 3.1, the  $V_{\text{fix}}^\sigma$  is defined with respect to the oracle in Assumption 2.

In Section 6.5.1, a dynamic data structure is given to efficiently compute  $\text{NextVar}(\sigma)$ .

With this construction of  $\text{NextVar}(\cdot)$ , the  $\text{MarginOverflow}$  subroutine is described in Algorithm 4.

Basically, a variable  $u$  is a good candidate for sampling if it currently has enough “freedom” (since  $u$  is not  $\sigma$ -fixed) and can “influence” the variables that we are trying to sample in the recursion (which are marked by  $\star$ ) through a chain of constraints in the simplification of  $\Phi$  under the current  $\sigma$ . Such variables are enumerated by  $\text{NextVar}(\sigma)$ .

The idea of Algorithm 4 is simple. In order to draw from the overflow distribution  $\mathcal{D}$  for a variable  $v \in V$ : if there is another candidate variable  $u = \text{NextVar}(\sigma)$  that still has enough freedom so its sampling might be easy, and also is relevant to the sampling at  $v$  or its ancestors, we try to sample  $u$ 's marginal value first (hopefully within its zone of local uniformity and compensated by a recursive call for drawing from its margin overflow); and if there is no such candidate variable to sample first, we finally draw from  $\mathcal{D}$  using the Bernoulli factory.

The following invariant is satisfied by the  $\text{MarginOverflow}$  subroutine called within the  $\text{MarginSample}$  subroutine and the  $\text{MarginOverflow}$  itself (formally proved in Lemma 5.4).

**Condition 3.8** (invariant for  $\text{MarginOverflow}$ ). *The following holds for the input tuple  $(\Phi, \sigma, v)$ :*

- $\Phi = (V, \mathcal{Q}, \mathcal{C})$  is a CSP formula,  $\sigma \in \mathcal{Q}^*$  is a feasible partial assignment, and  $v \in V$  is a variable;

---

**Algorithm 4:** MarginOverflow( $\Phi, \sigma, v$ )

---

**Input:** a CSP formula  $\Phi = (V, \mathcal{C})$ , a feasible partial assignment  $\sigma \in \mathcal{Q}^*$ , and a variable  $v \in V$ ;  
**Output:** a random  $x \in Q_v$  distributed as  $\mathcal{D} \triangleq \frac{1}{(1-q_v \cdot \theta_v)}(\mu_v^\sigma - \theta_v)$ ;

```
1  $u \leftarrow \text{NextVar}(\sigma)$  where  $\text{NextVar}(\sigma)$  is defined as in (9);
2 if  $u \neq \perp$  then
3   choose  $r \in [0, 1)$  uniformly at random;
4   if  $r < q_u \cdot \theta_u$  then           //  $r$  falls into the zone of local uniformity
5      $\sigma(u) \leftarrow$  the  $\lceil r/\theta_u \rceil$ -th value in  $Q_u$ ;
6   else                               //  $r$  falls into the zone of indecision
7      $\sigma(u) \leftarrow \text{MarginOverflow}(\Phi, \sigma_{u \leftarrow \star}, u)$ ;
8   /* Line 4 to Line 7 together draw  $\sigma(u)$  according to  $\mu_u^{\sigma_{u \leftarrow \star}}$  */
9   return MarginOverflow( $\Phi, \sigma, v$ );
9 else // All non- $\sigma$ -fixed variables are d/c. from  $v$  and ancestors.
10  sample a random  $x \in Q_v$  according to  $\mathcal{D} \triangleq \frac{1}{(1-q_v \cdot \theta_v)}(\mu_v^\sigma - \theta_v)$  using the Bernoulli factory in
    Appendix A that accesses RejectionSampling( $\Phi, \sigma, \{v\}$ ) as an oracle;
11  return  $x$ ;
```

---

- $\sigma(v) = \star$ ;
- $\mathbb{P}[-c \mid \sigma] \leq \alpha q$  for all  $c \in \mathcal{C}$ .

The following marginal lower bound follows from the “local uniformity” property (Corollary 4.3) in the same way as in Proposition 3.5 and is formally proved in Section 4.

**Proposition 3.9.** *Assuming Condition 3.8 for the input  $(\Phi, \sigma, v)$ , it holds that  $\min_{x \in Q_v} \mu_v^\sigma(x) \geq \theta_v + \zeta$  and for  $u = \text{NextVar}(\sigma)$ , if  $u \neq \perp$  then it also holds that  $\min_{x \in Q_u} \mu_u^\sigma(x) \geq \theta_u + \zeta$ .*

The Bernoulli factory used in Algorithm 4 is achieved by a combination of existing constructions (to be specific, the Bernoulli factory for subtraction in [NP05], composed with the linear Bernoulli factory in [Hub16] and the Bernoulli race in [DHKN17]), given access to an oracle for drawing from the marginal distribution  $\mu_v^\sigma$ , which is realized by RejectionSampling( $\Phi, \sigma, \{v\}$ ) in Algorithm 2. This is formally stated by the following lemma.

**Lemma 3.10** (correctness of Bernoulli factory). *Assuming Condition 3.8 for the input  $(\Phi, \sigma, v)$ , there is a Bernoulli factory accessing RejectionSampling( $\Phi, \sigma, \{v\}$ ) as an oracle that terminates with probability 1, and upon termination it returns a random  $x \in Q_v$  distributed as the  $\mathcal{D}$  defined in (8).*

The construction of the Bernoulli factory stated in above lemma is somehow standard, and is deferred to Appendix A, where Lemma 3.10 is proved and the efficiency of the Bernoulli factory is also analyzed.

#### 4. PRELIMINARY ON LOVÁSZ LOCAL LEMMA

The following is the asymmetric Lovász Local Lemma stated in the context of CSP.

**Theorem 4.1** (Erdős and Lovász [EL75]). *Given a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ , if the following holds*

$$(10) \quad \exists x \in (0, 1)^{\mathcal{C}} \quad \text{s.t.} \quad \forall c \in \mathcal{C} : \quad \mathbb{P}[-c] \leq x(c) \quad \prod_{\substack{c' \in \mathcal{C} \\ \text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset}} (1 - x(c')),$$

then

$$\mathbb{P} \left[ \bigwedge_{c \in \mathcal{C}} c \right] \geq \prod_{c \in \mathcal{C}} (1 - x(c)) > 0,$$

When the condition (10) is satisfied, the probability of any event in the uniform distribution  $\mu$  over all satisfying assignments can be well approximated by the probability of the event in the product distribution. This was observed in [HSS11]:

**Theorem 4.2** (Haeupler, Saha, and Srinivasan [HSS11]). *Given a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ , if (10) holds, then for any event  $A$  that is determined by the assignment on a subset of variables  $\text{vbl}(A) \subseteq V$ ,*

$$\Pr_{\mu}[A] = \mathbb{P}\left[A \mid \bigwedge_{c \in \mathcal{C}} c\right] \leq \mathbb{P}[A] \prod_{\substack{c \in \mathcal{C} \\ \text{vbl}(c) \cap \text{vbl}(A) \neq \emptyset}} (1 - x(c))^{-1},$$

where  $\mu$  denotes the uniform distribution over all satisfying assignments of  $\Phi$  and  $\mathbb{P}$  denotes the law of the uniform product distribution over  $\mathcal{Q}$ .

The following ‘‘local uniformity’’ property is a straightforward corollary to Theorem 4.2 by setting  $x(c) = ep$  for every  $c \in \mathcal{C}$  (and the lower bound is calculated by  $\mu_v(x) = 1 - \sum_{y \in \mathcal{Q}_v \setminus \{x\}} \mu_v(y)$ ).

**Corollary 4.3** (local uniformity). *Given a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$ , if  $ep\Delta < 1$ , then for any variable  $v \in V$  and any value  $x \in \mathcal{Q}_v$ , it holds that*

$$\frac{1}{q_v} - \left((1 - ep)^{-\Delta} - 1\right) \leq \mu_v(x) \leq \frac{1}{q_v} + \left((1 - ep)^{-\Delta} - 1\right).$$

The following corollary implied by the ‘‘local uniformity’’ property simultaneously proves Proposition 3.5 and Proposition 3.9. Recall  $\alpha$  defined in (6) and  $\theta_v, \zeta, \eta$  defined in (7).

**Corollary 4.4.** *For any CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  and any partial assignment  $\sigma \in \mathcal{Q}^*$ , if*

$$\forall c \in \mathcal{C}, \quad \mathbb{P}[\neg c \mid \sigma] \leq \alpha q,$$

then  $\sigma$  is feasible, and for any  $v \in V \setminus \Lambda(\sigma)$  and any  $x \in \mathcal{Q}_v$ ,

$$\theta_v + \zeta \leq \mu_v^{\sigma}(x) \leq \theta_v + 2\eta + \zeta.$$

*Proof.* Let  $\Phi^{\sigma} = (V^{\sigma}, \mathcal{Q}^{\sigma}, \mathcal{C}^{\sigma})$  be the simplification of  $\Phi$  over  $\sigma$ , where  $V^{\sigma} = V \setminus \Lambda(\sigma)$ . We have

$$\forall c \in \mathcal{C}, \quad \mathbb{P}_{\Phi^{\sigma}}[\neg c] = \mathbb{P}_{\Phi}[\neg c \mid \sigma] \leq \alpha q,$$

which means the simplified instance  $\Phi^{\sigma}$  has violation probability  $p_{\Phi^{\sigma}} \leq \alpha q$ . By our choice of  $\alpha$  in (6), we still have  $ep_{\Phi^{\sigma}}\Delta_{\Phi^{\sigma}} < 1$  where  $\Delta_{\Phi^{\sigma}} \leq \Delta$  is the constraint degree of simplified instance  $\Phi^{\sigma}$ . Then by Theorem 4.1,  $\Phi^{\sigma}$  is satisfiable, i.e.  $\sigma$  is feasible.

Note that the marginal distribution at  $v$  induced by the  $\mu_{\Phi^{\sigma}}$  over satisfying assignments of  $\Phi^{\sigma}$  is precisely  $\mu_v^{\sigma}$ . By Corollary 4.3, for any  $v \in V^{\sigma} = V \setminus \Lambda(\sigma)$  and any  $x \in \mathcal{Q}_v$ ,

$$\theta_v + \zeta = \frac{1}{q_v} - \eta = \frac{1}{q_v} - \left((1 - e\alpha q)^{-\Delta} - 1\right) \leq \mu_v^{\sigma}(x) \leq \frac{1}{q_v} + \left((1 - e\alpha q)^{-\Delta} - 1\right) = \frac{1}{q_v} + \eta = \theta_v + 2\eta + \zeta. \quad \square$$

## 5. CORRECTNESS OF SAMPLING

In this section, we prove the correctness of Algorithm 1. All theorems in this section assume the setting of parameters in (6) and (7), and the oracles in Assumption 1 and Assumption 2.

We show that our main sampling algorithm Algorithm 1 is correct.

**Theorem 5.1.** *On any input CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  satisfying (3), Algorithm 1 terminates with probability 1, and returns a uniform random satisfying assignment of  $\Phi$  upon termination.*

**Remark 5.2** (perfectness of sampling). Note that the sampling in above theorem is *perfect*: Algorithm 1 returns a sample that is distributed exactly as the uniform distribution  $\mu$  over all satisfying assignments of  $\Phi$ . Later in Section 6, the oracle assumed in Assumption 2 will be realized by a Monte Carlo routine, which will further generalize the sampling algorithm to assume nothing beyond an evaluation oracle, in a price of a bounded bias introduced to the sampling.

**Remark 5.3** (a weaker LLL condition). The LLL condition (3) is assumed mainly to guarantee the efficiency of the algorithm. Theorem 5.1 in fact holds under a much weaker LLL condition:

$$2e \cdot q^2 \cdot p \cdot \Delta < 1.$$

Under this condition, there exists such choices of parameters  $\alpha$  and  $\zeta$  that satisfy  $p < \alpha < \frac{1}{2eq^2\Delta}$  and  $0 < \zeta < \frac{1}{q} - (1 - e\alpha q)^{-\Delta} + 1$ . For any such choice of parameters, the same analysis persists and Algorithm 1 is as correct as claimed in Theorem 5.1.

The following lemma guarantees that the invariants in Condition 3.3 and Condition 3.8 are satisfied respectively by the inputs to Algorithm 3 and Algorithm 4.

**Lemma 5.4.** *During the execution of Algorithm 1 on a CSP formula  $\Phi = (V, Q, C)$  satisfying (3):*

- (1) *whenever  $\text{MarginSample}(\Phi, X, v)$  is called, Condition 3.3 is satisfied by its input  $(\Phi, X, v)$ ;*
- (2) *whenever  $\text{MarginOverflow}(\Phi, \sigma, v)$  is called, Condition 3.8 is satisfied by its input  $(\Phi, \sigma, v)$ .*

Before proving this lemma, we show that these invariants can already imply the correctness of  $\text{MarginSample}$ , which is critical for the correctness of the main sampling algorithm (Algorithm 1), because the correctness of  $\text{RejectionSampling}$  is standard (Theorem 3.4).

**Theorem 5.5.** *The following holds for Algorithm 3 and Algorithm 4:*

- (1) *Assuming Condition 3.3,  $\text{MarginSample}(\Phi, \sigma, v)$  terminates with probability 1, and it returns a random value  $x \in Q_v$  distributed as  $\mu_v^\sigma$  upon termination.*
- (2) *Assuming Condition 3.8,  $\text{MarginOverflow}(\Phi, \sigma, v)$  terminates with probability 1, and upon termination it returns a random value  $x \in Q_v$  distributed as the  $\mathcal{D} \triangleq \frac{\mu_v^\sigma - \theta_v}{1 - q_v \cdot \theta_v}$  defined in (8).*

*Proof.* We verify the correctness of  $\text{MarginOverflow}$  by a structural induction. Then the correctness of  $\text{MarginSample}$  follows straightforwardly.

Suppose that  $\text{MarginOverflow}$  is run on input  $(\Phi, \sigma, v)$  satisfying Condition 3.8.

The induction basis is when  $\text{NextVar}(\sigma) = \perp$ , in which case no further recursive calls to the  $\text{MarginOverflow}$  are incurred. In this case, due to the correctness of the Bernoulli factory stated in Lemma 3.10 under Condition 3.8,  $\text{MarginOverflow}(\Phi, \sigma, v)$  terminates with probability 1 and returns a random value  $x \in Q_v$  distributed as  $\mathcal{D}$ .

For the induction step, we assume that  $\text{NextVar}(\sigma) = u \in V$ . By the induction hypothesis, the recursive calls to  $\text{MarginOverflow}$  at Line 7 and Line 8 in Algorithm 4 terminate with probability 1. All other computations are finite. By induction,  $\text{MarginOverflow}(\Phi, \sigma, v)$  terminates with probability 1.

We then verify the correctness of sampling. Let  $W$  denote the value of  $\sigma(u)$  generated in Lines 3-7 of in Algorithm 4. It is easy to verify that for every  $a \in Q_u$ ,

$$(11) \quad \begin{aligned} \Pr [W = a] &= \Pr [r < q_u \cdot \theta_u] \cdot \frac{1}{q_u} + \Pr [r \geq q_u \cdot \theta_u] \cdot \Pr [\text{MarginOverflow}(\Phi, \sigma_{u \leftarrow \star}, u) = a] \\ &= \theta_u + \mu_u^\sigma(a) - \theta_u = \mu_u^\sigma(a), \end{aligned}$$

where the second equality is due to the induction hypothesis (I.H.). Thus, for every  $a \in Q_u$ ,

$$\begin{aligned} \Pr [\text{MarginOverflow}(\Phi, \sigma, v) = a] &= \sum_{b \in Q_u} (\Pr [W = b] \cdot \Pr [\text{MarginOverflow}(\Phi, \sigma_{u \leftarrow b}, v) = a]) \\ \text{(by (11) and I.H.)} \quad &= \sum_{b \in Q_u} \left( \mu_u^\sigma(b) \cdot \frac{\mu_v^{\sigma_{u \leftarrow b}}(a) - \theta_v}{1 - q_v \cdot \theta_v} \right) \\ &= \frac{\mu_v^\sigma(a) - \theta_v}{1 - q_v \cdot \theta_v}, \end{aligned}$$

which means that the value returned by  $\text{MarginOverflow}(\Phi, \sigma, v)$  is distributed as  $\mathcal{D}$ . This finishes the induction and proves the correctness of  $\text{MarginOverflow}$  assuming Condition 3.8.

It is then straightforward to verify the correctness of  $\text{MarginSample}$  under Condition 3.3. Let  $(\Phi, \sigma, v)$  be an arbitrary input to  $\text{MarginSample}$  satisfying Condition 3.3. It is easy to verify that  $(\Phi, \sigma_{v \leftarrow \star}, v)$  satisfies Condition 3.8, and hence the correctness of  $\text{MarginOverflow}$  can apply. Therefore,  $\text{MarginSample}(\Phi, \sigma, v)$  terminates with probability 1 and for every  $a \in Q_v$ ,

$$\Pr [\text{MarginSample}(\Phi, \sigma, v) = a]$$

$$\begin{aligned}
&= \Pr [r < q_v \cdot \theta_v] \cdot \frac{1}{q_v} + \Pr [r \geq q_v \cdot \theta_v] \cdot \Pr [\text{MarginOverflow}(\Phi, \sigma_{u \leftarrow \star}, v) = a] \\
&= \theta_v + \mu_v^\sigma(a) - \theta_v \\
&= \mu_v^\sigma(a).
\end{aligned}$$

This shows the termination and correctness of MarginSample.  $\square$

We then verify the invariant conditions claimed in Lemma 5.4. Before that, we formally define the sequence of partial assignments that evolve in Algorithm 1.

**Definition 5.6** (partial assignments in Algorithm 1). Let  $X^0, X^1, \dots, X^n \in \mathcal{Q}^*$  denote the sequence of partial assignments, where  $X^0 = \star^V$  and for every  $1 \leq i \leq n$ ,  $X^i$  is the partial assignments  $X$  in Algorithm 1 after the  $i$ -th iteration of the **for** loop in Lines 1-4.

**Fact 5.7.** For each  $1 \leq i \leq n$ , either  $v_i$  is  $X^{i-1}$ -fixed, in which case  $X^i = X^{i-1}$ , or otherwise, in which case  $X^i$  extends  $X^{i-1}$  by assigning  $v_i$  a value in  $Q_{v_i}$ . Consequently, for each  $1 \leq i \leq n$ , if  $v_i$  is not  $X^{i-1}$ -fixed, then  $X^*(v_i) = X^i(v_i)$ , where  $X^*$  denotes the output of Algorithm 1.

**Lemma 5.8.** For the  $X^0, X^1, \dots, X^n$  in Definition 5.6, it holds for all  $0 \leq i \leq n$  that  $X^i$  is feasible and

$$(12) \quad \forall c \in \mathcal{C}, \quad \mathbb{P}[\neg c \mid X^i] < \alpha q.$$

*Proof.* We only need to prove (12). Then the feasibility of  $X^i$  follows from Corollary 4.4.

Fix any  $c \in \mathcal{C}$ . Recall that assuming the LLL condition (3), we have  $\alpha > p$  by (6). Then we have

$$\mathbb{P}[\neg c \mid X^0] = \mathbb{P}[\neg c \mid \star^V] = \mathbb{P}[\neg c] \leq p < \alpha.$$

Suppose  $i^*$  to be the smallest  $0 \leq i \leq n$  such that  $\mathbb{P}[\neg c \mid X^i] \geq \alpha$ . By  $\mathbb{P}[\neg c \mid X^0] < \alpha$ , we have  $i^* \geq 1$ . Then  $\mathbb{P}[\neg c \mid X^{i^*-1}] \leq \alpha$ , which means  $X^{i^*} \neq X^{i^*-1}$ . Combining with Definition 5.6 and Algorithm 1, we have  $X^{i^*}$  extends  $X^{i^*-1}$  by assigning  $v_{i^*}$  some value in  $Q_{v_{i^*}}$ . Therefore,

$$(13) \quad \mathbb{P}[\neg c \mid X^{i^*}] \leq \frac{\mathbb{P}[\neg c \mid X^{i^*-1}]}{\min_{x \in Q_{v_{i^*}}} \mathbb{P}[v_{i^*} = x \mid X^{i^*-1}]} < \alpha |Q_{v_{i^*}}| \leq \alpha q.$$

Thus,  $\alpha \leq \mathbb{P}[\neg c \mid X^{i^*}] < \alpha q$ . Combining with Definition 3.1, we have all variables in  $\text{vbl}(c)$  are  $X^{i^*}$ -fixed, and thus the variables in  $\text{vbl}(c)$  will stay unchanged in Algorithm 1. Therefore, if  $i^* < n$ , we have  $X^{i^*+1}(v) = X^{i^*}(v)$  for each  $v \in \text{vbl}(c)$  and then  $\alpha \leq \mathbb{P}[\neg c \mid X^{i^*+1}] = \mathbb{P}[\neg c \mid X^{i^*}] < \alpha q$ . Iteratively, one can show that

$$\forall i^* \leq i \leq n, \quad \alpha \leq \mathbb{P}[\neg c \mid X^i] < \alpha q.$$

In addition, by  $i^*$  is the smallest  $0 \leq i \leq n$  such that  $\mathbb{P}[\neg c \mid X^i] \geq \alpha$ , we have

$$\forall 0 \leq i < i^*, \quad \mathbb{P}[\neg c \mid X^i] < \alpha < \alpha q.$$

Then (12) is proved.  $\square$

The invariant of Condition 3.3 for MarginSample stated in Lemma 5.4-(1) follows easily from Lemma 5.8. To prove the invariant of Condition 3.8 for MarginOverflow, we show the following.

**Lemma 5.9.** Assume Condition 3.8 for  $(\Phi, \sigma, v)$ . For any  $u \in V$ , if  $u$  is not  $\sigma$ -fixed, then  $(\Phi, \sigma_{u \leftarrow a}, v)$  and  $(\Phi, \sigma_{u \leftarrow \star}, u)$  satisfy Condition 3.8 for any  $a \in Q_u \cup \{\star\}$ .

*Proof.* The proofs for  $(\Phi, \sigma_{u \leftarrow \star}, u)$  and  $(\Phi, \sigma_{u \leftarrow a}, v)$  are similar. Thus, it suffices to that prove  $(\Phi, \sigma_{u \leftarrow a}, v)$  satisfies Condition 3.8 for any  $a \in Q_u \cup \{\star\}$ .

The feasibility of  $\sigma_{u \leftarrow a}$  follows from Corollary 4.4. Meanwhile, we also have  $\sigma_{u \leftarrow \star}(v) = \star$ . In the next, we prove that  $\mathbb{P}[\neg c \mid \sigma_{u \leftarrow \star}] \leq \alpha q$  for all  $c \in \mathcal{C}$ . If  $a = \star$ ,  $\mathbb{P}[\neg c \mid \sigma_{u \leftarrow \star}] \leq \alpha q$  holds trivially, because as a not  $\sigma$ -fixed variable,  $u$  must have  $\sigma(u) = \star$ , and hence changing  $\sigma$  to  $\sigma_{u \leftarrow \star}$  does not change the probability of any event conditioning on  $\sigma$ . And for the case that  $a \in Q_u$ : if  $c$  is  $\sigma$ -frozen then  $u \notin \text{vbl}(c)$  since  $u$  is not  $\sigma$ -fixed, and hence we have  $\mathbb{P}[\neg c \mid \sigma_{u \leftarrow a}] = \mathbb{P}[\neg c \mid \sigma] \leq \alpha q$ , where the

inequality is by that  $(\Phi, \sigma, v)$  satisfies Condition 3.8; and if otherwise  $c$  is not  $\sigma$ -frozen, which means  $\mathbb{P}[-c \mid \sigma] \leq \alpha$ , and hence as calculated in (13), we have

$$\mathbb{P}[-c \mid \sigma_{u \leftarrow a}] \leq |Q_u| \mathbb{P}[-c \mid \sigma] \leq \alpha q,$$

which finishes the proof.  $\square$

The invariant of Condition 3.8 for MarginOverflow stated in Lemma 5.4-(2) follows from Lemmas 5.4-(1) and 5.9. Because by  $(\Phi, \sigma, v)$  satisfies Condition 3.3, we have  $(\Phi, \sigma_{v \leftarrow \star}, v)$  satisfies Condition 3.8. In addition, during the execution of MarginOverflow( $\Phi, \tau, v$ ) where  $\tau = \sigma_{v \leftarrow \star}$ , the algorithm will only change an input partial assignment  $\tau$  to  $\tau_{u \leftarrow a}$  for those vertices  $u$  that are not  $\tau$ -fixed  $u$  and for  $a \in Q_u \cup \{\star\}$ . Lemma 5.4 is proved.

Combining Lemma 5.4 and Theorem 5.5, we prove the correctness of MarginSample (Algorithm 3), assuming the LLL condition in (3) for the input CSP in the main algorithm (Algorithm 1).

The correctness of RejectionSampling (Algorithm 2) has already been established in Theorem 3.4 (which is standard).

The correctness of the main sampling algorithm (Algorithm 1) then follows from the correctness of these two main subroutines. Note that this is not trivial because in Algorithm 1, the variables are chosen to draw from their marginal distributions adaptive to randomness. We then formally prove that such being adaptive to randomness does not affect the correctness of sampling.

*Proof of Theorem 5.1.* By Lemmas 3.4 and Lemma 3.10, Algorithm 1 terminates with probability 1.

Let  $X^0, X^1, \dots, X^n$  be the sequence of partial assignments defined in Definition 5.6. Let  $X^*$  denote the output of Algorithm 1. We then show that, for every  $\sigma \in \Omega$ ,  $\Pr[X^* = \sigma] = \mu(\sigma)$ .

Fix an arbitrary satisfying assignment  $\sigma \in \Omega$ . We further define a sequence of partial assignments  $\sigma^0, \sigma^1, \dots, \sigma^n$  as follows. Let  $\sigma^0 = \star^V$ . For each  $1 \leq i \leq n$ , if  $v_i$  is  $\sigma^{i-1}$ -fixed, let  $\sigma^i = \sigma^{i-1}$ ; otherwise, let  $\sigma^i = \sigma_{v_i \leftarrow \sigma(v_i)}^{i-1}$ . We claim that for every  $0 \leq i \leq n$ ,

$$(14) \quad \prod_{j=1}^i \Pr[X^j = \sigma^j \mid X^{j-1} = \sigma^{j-1}] = \mu_{\Lambda(\sigma^i)}(\sigma_{\Lambda(\sigma^i)}),$$

with convention that both sides equal to 1 for  $i = 0$ .

We then prove this claim by an induction on  $i$ . The basis with  $i = 0$  holds by the convention.

For the induction step, we consider  $i \geq 1$ . By the consistency of the oracle in Assumption 2, if  $X^{i-1} = \sigma^{i-1}$ , then  $v_i$  can only be simultaneously fixed or non-fixed in both  $X^{i-1}$  and  $\sigma^{i-1}$ .

- If  $v_i$  is  $\sigma^{i-1}$ -fixed, then  $\sigma^i = \sigma^{i-1}$ , and by Lines 3-4 of Algorithm 1, we have

$$\Pr[X^i = \sigma^i \mid X^{i-1} = \sigma^{i-1}] = 1.$$

Thus,

$$\begin{aligned} \prod_{j=1}^i \Pr[X^j = \sigma^j \mid X^{j-1} = \sigma^{j-1}] &= \prod_{j=1}^{i-1} \Pr[X^j = \sigma^j \mid X^{j-1} = \sigma^{j-1}] \\ &\quad \text{(by I.H.)} = \mu_{\Lambda(\sigma^{i-1})}(\sigma_{\Lambda(\sigma^{i-1})}) \\ &\quad \text{(since } \sigma^i = \sigma^{i-1}) = \mu_{\Lambda(\sigma^i)}(\sigma_{\Lambda(\sigma^i)}). \end{aligned}$$

- If  $v_i$  is not  $\sigma^{i-1}$ -fixed, then by the correctness of MarginSample (guaranteed by Lemma 5.4 and Theorem 5.5), we have

$$\Pr[X^i = \sigma^i \mid X^{i-1} = \sigma^{i-1}] = \mu_{v_i}^{\sigma^{i-1}}(\sigma(v_i)).$$

Thus, we have

$$\begin{aligned} \prod_{j=1}^i \Pr[X^j = \sigma^j \mid X^{j-1} = \sigma^{j-1}] &= \mu_{v_i}^{\sigma^{i-1}}(\sigma(v_i)) \cdot \prod_{j=1}^{i-1} \Pr[X^j = \sigma^j \mid X^{j-1} = \sigma^{j-1}] \\ &\quad \text{(by I.H.)} = \mu_{v_i}^{\sigma^{i-1}}(\sigma(v_i)) \cdot \mu_{\Lambda(\sigma^{i-1})}(\sigma_{\Lambda(\sigma^{i-1})}) \end{aligned}$$

$$\text{(chain rule)} \quad = \mu_{\Lambda(\sigma^i)}(\sigma_{\Lambda(\sigma^i)}).$$

This finishes the induction. The claim in (14) is proved.

Observe that the sequence  $X_0, X_1, \dots, X_n, X^*$  is a Markov chain, where the last step  $X^*$  is constructed from  $X^n$  by RejectionSampling. Suppose that event  $X^* = \sigma$ . By Fact 5.7 we have  $X^i(v_i) = \sigma(v_i)$  if  $v_i$  is not  $X^{i-1}$ -fixed. Therefore according to Definition 5.6, we have that if  $v_i$  is  $X^{i-1}$ -fixed, then  $X^i = X^{i-1}$ ; and if otherwise  $X^i = X_{v_i \leftarrow \sigma(v_i)}^{i-1}$ . Thus, given that  $X^* = \sigma$  occurs, by the consistency of the oracle in Assumption 2, one can verify that  $X^i = \sigma^i$  for all  $0 \leq i \leq n$ . Thus,

$$\begin{aligned} \Pr[X^* = \sigma] &= \Pr\left[(X^* = \sigma) \wedge \left(\bigwedge_{1 \leq i \leq n} (X^i = \sigma^i)\right)\right] \\ \text{(chain rule)} \quad &= \Pr[X^* = \sigma \mid \forall 1 \leq i \leq n, X^i = \sigma^i] \cdot \prod_{i=1}^n \Pr[X^i = \sigma^i \mid \forall 0 \leq j < i, X^j = \sigma^j] \\ \text{(Markov property)} \quad &= \Pr[X^* = \sigma \mid X^n = \sigma^n] \cdot \prod_{i=1}^n \Pr[X^i = \sigma^i \mid X^{i-1} = \sigma^{i-1}] \\ \text{(by (14))} \quad &= \Pr[X^* = \sigma \mid X^n = \sigma^n] \cdot \mu_{\Lambda(\sigma^n)}(\sigma_{\Lambda(\sigma^n)}) \\ \text{(by Theorem 3.4)} \quad &= \mu_{V \setminus \Lambda(\sigma^n)}^{\sigma^n}(\sigma_{V \setminus \Lambda(\sigma^n)}) \cdot \mu_{\Lambda(\sigma^n)}(\sigma_{\Lambda(\sigma^n)}) \\ \text{(chain rule)} \quad &= \mu(\sigma). \quad \square \end{aligned}$$

## 6. EFFICIENCY OF SAMPLING

In this section, we show the efficiency of Algorithm 1 under the LLL condition in (3).

Algorithm 1 assumes accesses to the following oracles for a class of constraints  $\mathcal{C}$ , both of which receive as input a constraint  $c \in \mathcal{C}$  and a partial assignment  $\sigma \in \mathcal{Q}^*$  upon queries:

- $\text{Eval}(c, \sigma)$ : the evaluation oracle in Assumption 1, which decides whether  $\mathbb{P}[c \mid \sigma] = 1$ , that is, whether  $c$  is already satisfied by  $\sigma$ ;
- $\text{Frozen}(c, \sigma)$ : the oracle for frozen decision in Assumption 2, which distinguishes between the two cases  $\mathbb{P}[-c \mid \sigma] > \alpha$  and  $\mathbb{P}[-c \mid \sigma] < 0.99\alpha$ , where  $\alpha$  is the threshold defined in (6), and answers arbitrarily and consistently if otherwise.

The complexity of our sampling algorithm is measured in terms of the queries to the two oracles  $\text{Eval}(\cdot)$  and  $\text{Frozen}(\cdot)$ , and the computation costs. We prove the following theorem.

**Theorem 6.1.** *Given as input a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  satisfying (3), Algorithm 1 in expectation costs  $O(q^2 k^2 \Delta^9 n)$  queries to  $\text{Eval}(\cdot)$ ,  $O(k \Delta^6 n)$  queries to  $\text{Frozen}(\cdot)$ , and  $O(q^3 k^3 \Delta^9 n)$  in computation.*

Together with the correctness of Algorithm 1 stated in Theorem 5.1, this proves the main theorem for perfect sampling (Theorem 1.2), since any query to the oracle  $\text{Frozen}(\cdot)$  can be resolved in  $\text{poly}(q, k)$  time assuming the FPTAS for violation probability in the condition of Theorem 1.2.

**Remark 6.2 (Monte Carlo realization of frozen decision).** The oracle  $\text{Frozen}(\cdot)$  can be realized probabilistically through the Monte Carlo method. Upon each query on a constraint  $c$  and a partial assignment  $\sigma$ , the two extreme cases  $\mathbb{P}[-c \mid \sigma] > \alpha$  and  $\mathbb{P}[-c \mid \sigma] < 0.99\alpha$  can be distinguished with high probability  $(1 - \delta)$  by independently testing for  $O(\frac{1}{\alpha} \log \frac{1}{\delta})$  times whether the constraint  $c$  is satisfied by a randomly generated assignment over  $\text{vbl}(c)$  consistent with  $\sigma$ . We further apply a memoization to guarantee the consistency of the oracle as required in Assumption 2. The resulting algorithm is called Algorithm 1' and is formally described in Section 6.7.

This Monte Carlo realization of the  $\text{Frozen}(\cdot)$  oracle introduces a bounded bias to the result of sampling and turns the perfect sampler in Theorem 6.1 to an approximate sampler Algorithm 1', which no longer assumes any nontrivial machinery beyond evaluating constraints.

**Theorem 6.3.** *Given as input an  $\varepsilon \in (0, 1)$  and a CSP formula  $\Phi$  satisfying (3), Algorithm 1' in expectation costs  $O(q^2 k^2 \Delta^9 n \log(\frac{\Delta n}{\varepsilon}))$  queries to  $\text{Eval}(\cdot)$  and  $O(q^3 k^3 \Delta^9 n \log(\frac{\Delta n}{\varepsilon}))$  in computation, and outputs within  $\varepsilon$  total variation distance from the output of Algorithm 1 on input  $\Phi$ .*



Together with the correctness of Algorithm 1 stated in Theorem 5.1, this proves the main theorem (Theorem 1.1) of the paper.

**A notation for complexity bound:** Throughout the section, we adopt the following abstract notation for any complexity bound. A complexity bound is expressed as a formal bi-variate linear function:

$$(15) \quad t(\underline{x}, \underline{y}) = \alpha \cdot \underline{x} + \beta \cdot \underline{y} + c,$$

where  $\alpha$  represents the number of queries to  $\text{Eval}(\cdot)$ ,  $\beta$  represents the number of queries to  $\text{Frozen}(\cdot)$ , and  $\gamma$  represents the computation costs.

For examples, The complexity bounds in Theorems 6.1 and 6.3 are thus expressed respectively as:

$$O\left((q^2 k^2 \Delta^9 n) \cdot \underline{x} + (k \Delta^6 n) \cdot \underline{y} + q^3 k^3 \Delta^9 n\right) \text{ and } O\left(\left(q^2 k^2 \Delta^9 n \log\left(\frac{\Delta n}{\varepsilon}\right)\right) \cdot \underline{x} + q^3 k^3 \Delta^9 n \log\left(\frac{\Delta n}{\varepsilon}\right)\right).$$

We remark that such expression is only for notational convenience, because we want to handle three different complexity measures simultaneously in the analyses. Throughout our analyses, only linear calculations will be applied to such functions  $t(\underline{x}, \underline{y})$ . We further express:

$$\alpha \cdot \underline{x} + \beta \cdot \underline{y} + \gamma \leq \alpha' \cdot \underline{x} + \beta' \cdot \underline{y} + \gamma' \iff \alpha \leq \alpha' \wedge \beta \leq \beta' \wedge \gamma \leq \gamma'.$$

And we write  $t(0, 0)$  for the constant term  $\gamma$  in (15), which stands for the computation cost.

**6.1. Input model and data structure.** Besides being accessed through the two oracles  $\text{Eval}(\cdot)$  and  $\text{Frozen}(\cdot)$ , the input CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  is presented to the algorithm as follows:

- The variables in  $V = \{v_1, v_2, \dots, v_n\}$  and constraints  $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$  can be randomly accessed by their indices  $i \in [n]$  and  $j \in [m]$ .
- Given any  $c \in \mathcal{C}$ , the  $\text{vbl}(c)$  can be retrieved in time  $O(k)$ ; given any  $v \in V$ , the set of constraints  $c$  with  $v \in \text{vbl}(c)$  can be retrieved in time  $O(\Delta)$ ; given any  $c \in \mathcal{C}$ , the set of dependent  $c' \in \mathcal{C}$  with  $\text{vbl}(c')$  intersecting  $\text{vbl}(c)$  can be retrieved within time  $O(\Delta)$ .

These requirements can be met by representing the CSP  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  in its bipartite incidence graph and also the dependency graph, both using the adjacency linked list data structures.

The partial assignment  $\sigma \in \mathcal{Q}^*$  is maintained by the algorithm in such a way that passing  $\sigma$  to function as its argument takes  $O(1)$  time. This can be resolved by storing  $\sigma$  globally as an array of  $|V|$  stacks and passing a pointer to this array when  $\sigma$  is passed as a function argument, such that whenever a value  $x$  is assigned to  $\sigma(v)$ ,  $x$  is pushed into the stack associated to  $v$ ; and when a function returns it pops the stacks associated to those variables that it has updated in the current level of recursion.

The partial assignment  $\sigma \in \mathcal{Q}^*$  also keeps a linked list of the variables currently set as  $\star$ .

**6.2. The recursive cost tree (RCT).** A crucial step for proving Theorem 6.1 is the analysis of the  $\text{MarginSample}$  (Algorithm 3), which calls to the recursive subroutine  $\text{MarginOverflow}$  (Algorithm 4).

Consider an input  $(\Phi, \sigma, v)$  satisfying Condition 3.3 such that  $\text{MarginSample}(\Phi, \sigma, v)$  is well-defined. Our goal is to upper bound the following complexity.

**Definition 6.4.** Let  $\bar{t}_{\text{MS}}(\Phi, \sigma, v)$  denote the expected cost of  $\text{MarginSample}(\Phi, \sigma, v)$  (Algorithm 3).

There are two nontrivial tasks involved in computing the  $\text{MarginSample}(\Phi, \sigma, v)$ : computing of the  $\text{NextVar}(\sigma)$  and the Bernoulli factory, both of which are in the  $\text{MarginOverflow}$  (Algorithm 4).

**Definition 6.5.** Let  $t_{\text{var}}(\sigma)$  denote the cost for deterministically computing  $\text{NextVar}(\sigma)$  defined in (9). Let  $\bar{t}_{\text{BF}}(\sigma)$  be the expected cost for the Bernoulli factory in Line 10 of Algorithm 4 in the worst case of  $v$  such that Condition 3.8 is satisfied, if there exists such a  $v$ ; and let  $\bar{t}_{\text{BF}}(\sigma) = 0$ , if no such  $v$  exists.

The  $\bar{t}_{\text{BF}}(\sigma)$  upper bounds the expected cost for the Bernoulli factory on well-defined input  $(\Phi, \sigma, v)$ .

The above complexity bounds  $\bar{t}_{\text{MS}}(\Phi, \sigma, v)$ ,  $t_{\text{var}}(\sigma)$ , and  $\bar{t}_{\text{BF}}(\sigma)$  are all expressed in the form of (15). The concrete bounds for  $t_{\text{var}}(\sigma)$  and  $\bar{t}_{\text{BF}}(\sigma)$  are proved respectively in Sections 6.5.1 and 6.5.2.

We first introduce a combinatorial structure that relates  $\bar{t}_{\text{MS}}(\Phi, \sigma, v)$  to  $t_{\text{var}}(\sigma)$  and  $\bar{t}_{\text{BF}}(\sigma)$ . For each  $v \in V$ , we further define  $\mathcal{Q}_v^* \triangleq \mathcal{Q}_v \cup \{\star\}$  as an extended domain for accessment.

**Definition 6.6** (recursive cost tree). For any  $\sigma \in \mathcal{Q}^*$ , let  $\mathcal{T}_\sigma = (T_\sigma, \rho_\sigma)$ , where  $T_\sigma$  is a rooted tree with nodes  $V(T_\sigma) \subseteq \mathcal{Q}^*$  and  $\rho_\sigma : V(T_\sigma) \rightarrow [0, 1]$  is a labeling of nodes in  $T_\sigma$ , be constructed as:

- (1) The root of  $T_\sigma$  is  $\sigma$ , with  $\rho_\sigma(\sigma) = 1$  and depth of  $\sigma$  being 0;
- (2) for  $i = 0, 1, \dots$ : for all nodes  $\tau \in V(T_\sigma)$  of depth  $i$  in the current  $T_\sigma$ ,
  - (a) if  $\text{NextVar}(\tau) = \perp$ , then leave  $\tau$  as a leaf node in  $T_\sigma$ ;
  - (b) otherwise, supposed  $u = \text{NextVar}(\tau)$ , append  $\{\tau_{u \leftarrow x} \mid x \in Q_u \cup \{\star\}\}$  as the  $q_u + 1$  children to the node  $\tau$  in  $T_\sigma$ , and label them as:

$$\forall x \in Q_u \cup \{\star\}, \quad \rho_\sigma(\tau_{u \leftarrow x}) = \begin{cases} (1 - q_u \cdot \theta_u) \rho_\sigma(\tau) & \text{if } x = \star, \\ \mu_u^\sigma(x) \cdot \rho_\sigma(\tau) & \text{if } x \in Q_u. \end{cases}$$

The resulting  $\mathcal{T}_\sigma = (T_\sigma, \rho_\sigma)$  is called the *recursive cost tree (RCT)* rooted at  $\sigma$ .

Define the following function  $\lambda(\cdot)$  on RCTs  $\mathcal{T}_\sigma = (T_\sigma, \rho_\sigma)$ :

$$(16) \quad \lambda(\mathcal{T}_\sigma) \triangleq \sum_{\tau \in V(T_\sigma)} (\rho_\sigma(\tau) \cdot t_{\text{var}}(\tau)) + \sum_{\text{leaves } \tau \text{ in } T_\sigma} (\rho_\sigma(\tau) \cdot \bar{t}_{\text{BF}}(\tau)).$$

Note that  $\lambda(\mathcal{T}_\sigma)$  is expressed in the form of (15) as the  $t_{\text{var}}$  and  $\bar{t}_{\text{BF}}$ .

The expected complexity of MarginSample is bounded through this function  $\lambda(\mathcal{T}_\sigma)$ .

**Lemma 6.7.** *For any input  $(\Phi, \sigma, v)$  satisfying Condition 3.3, it holds for  $\sigma^* = \sigma_{v \leftarrow \star}$  that*

$$\bar{t}_{\text{MS}}(\Phi, \sigma, v) \leq (1 - q_v \cdot \theta_v)(\lambda(\mathcal{T}_{\sigma^*}) + \lambda(\mathcal{T}_{\sigma^*})(0, 0)) + O(1),$$

where  $\lambda(\mathcal{T}_{\sigma^*})(0, 0)$  is the constant term in  $\lambda(\mathcal{T}_{\sigma^*})$  (standing for the computation cost as in (15)).

In the following, we prove Lemma 6.7. The following recursive relation for RCT is easy to verify.

**Proposition 6.8.** *Let  $\sigma \in \mathcal{Q}^*$  and  $u = \text{NextVar}(\sigma)$ . If  $u \neq \perp$ , then*

$$\lambda(\mathcal{T}_\sigma) = t_{\text{var}}(\sigma) + (1 - q_u \cdot \theta_u) \lambda(\mathcal{T}_{\sigma_{u \leftarrow \star}}) + \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot \lambda(\mathcal{T}_{\sigma_{u \leftarrow x}})).$$

Then we show that the complexity upper bound in Lemma 6.7 holds for the MarginOverflow.

**Lemma 6.9.** *Let  $(\Phi, \sigma, v)$  be the input to MarginOverflow (Algorithm 4) satisfying Condition 3.8, and let  $\bar{t}_{\text{MO}}(\Phi, \sigma, v)$  denote the expected cost of MarginOverflow( $\Phi, \sigma, v$ ). It holds that*

$$\bar{t}_{\text{MO}}(\Phi, \sigma, v) \leq \lambda(\mathcal{T}_\sigma) + O(\lambda(\mathcal{T}_\sigma)(0, 0)),$$

where  $\lambda(\mathcal{T}_\sigma)(0, 0)$  is the constant term in  $\lambda(\mathcal{T}_\sigma)$ .

*Proof.* For simplicity, for any partial assignment  $\sigma \in \mathcal{Q}^*$ , we let  $\gamma(\sigma) = \lambda(\mathcal{T}_\sigma)(0, 0)$  denote the constant term in  $\lambda(\mathcal{T}_\sigma)$ . Let  $C > 0$  denote the constant computation cost that dominates the costs for argument passing and all computations in Lines 2-6 of MarginOverflow( $\Phi, \sigma, v$ ). It suffices to show that

$$\bar{t}_{\text{MO}}(\Phi, \sigma, v) \leq \lambda(\mathcal{T}_\sigma) + C \cdot \gamma(\sigma).$$

We prove this by an induction on the structure of RCT.

The base case is when  $T_\sigma$  is just a single root, in which case  $\text{NextVar}(\sigma) = \perp$ , and by Definition 6.6,

$$\lambda(\mathcal{T}_\sigma) = \rho_\sigma(\sigma) \cdot t_{\text{var}}(\sigma) + \rho_\sigma(\sigma) \cdot \bar{t}_{\text{BF}}(\sigma) = t_{\text{var}}(\sigma) + \bar{t}_{\text{BF}}(\sigma).$$

Also if  $\text{NextVar}(\sigma) = \perp$ , the condition in Line 2 of MarginOverflow( $\Phi, \sigma, v$ ) is unsatisfied and

$$\bar{t}_{\text{MO}}(\Phi, \sigma, v) \leq t_{\text{var}}(\sigma) + \mathbb{E}[T_{\text{BFS}}(\Phi, \sigma, v)] \leq t_{\text{var}}(\sigma) + \bar{t}_{\text{BF}}(\sigma) + C,$$

where  $T_{\text{BFS}}(\Phi, \sigma, v)$  represents the cost of the Bernoulli factory in Line 10 of Algorithm 4, and by Definition 6.5, it holds that  $\bar{t}_{\text{BF}}(\sigma) \geq \mathbb{E}[T_{\text{BFS}}(\Phi, \sigma, v)]$  for all such  $v$  that  $(\Phi, \sigma, v)$  satisfies Condition 3.8. The base case is proved.

For the induction step, we assume that  $T_\sigma$  is a tree of depth  $> 0$ . Thus by Definition 6.6,  $\text{NextVar}(\sigma) = u \neq \perp$  for some  $u \in V$ . According to Lines 3-7 of MarginOverflow( $\Phi, \sigma, v$ ), one can verify that for every  $x \in Q_u$ , the probability that  $\sigma(u) = x$  upon Line 8 is

$$\Pr[r < q_u \cdot \theta_u] \cdot \frac{1}{q_u} + \Pr[r \geq q_u \cdot \theta_u] \cdot \Pr[\text{MarginOverflow}(\Phi, \sigma_{u \leftarrow \star}, u) = x] = \mu_u^\sigma(x),$$

where the first equality is due to the correctness of MarginOverflow guaranteed in Theorem 5.5.

By the law of total expectation,

$$(17) \quad \bar{t}_{\text{MO}}(\Phi, \sigma, v) = t_{\text{var}}(\sigma) + (1 - q_u \cdot \theta_u) \cdot \bar{t}_{\text{MO}}(\Phi, \sigma_{u \leftarrow \star}, u) + \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot \bar{t}_{\text{MO}}(\Phi, \sigma_{u \leftarrow x}, v)) + C.$$

Note that by Item 2b in Definition 6.6, for each  $x \in Q_u \cup \{\star\}$ , the subtree in  $T_\sigma$  rooted by  $\sigma_{u \leftarrow x}$  is precisely the  $T_\tau$  in the RCT  $\mathcal{T}_\tau = (T_\tau, \rho_\tau)$  rooted at  $\tau = \sigma_{u \leftarrow x}$ . By Lemma 5.9, Condition 3.8 is still satisfied by  $(\Phi, \sigma_{u \leftarrow x}, v)$ . Thus, by induction hypothesis,

$$\bar{t}_{\text{MO}}(\Phi, \sigma_{u \leftarrow x}, u) \leq \lambda(\mathcal{T}_{\sigma_{u \leftarrow x}}) + C \cdot \gamma(\sigma_{u \leftarrow x}),$$

where  $\gamma(\sigma_{u \leftarrow x}) = \lambda(\mathcal{T}_{\gamma(\sigma_{u \leftarrow x})})(0, 0)$  represents the constant term in  $\lambda(\mathcal{T}_{\gamma(\sigma_{u \leftarrow x})})$ . Combined with (17),

$$\begin{aligned} \bar{t}_{\text{MO}}(\Phi, \sigma, v) &\leq t_{\text{var}}(\sigma) + (1 - q_u \cdot \theta_u) (\lambda(\mathcal{T}_{\sigma_{u \leftarrow \star}}) + C \cdot \gamma(\sigma_{u \leftarrow \star})) \\ &\quad + \sum_{x \in Q_u} (\mu_u^\sigma(x) (\lambda(\mathcal{T}_{\sigma_{u \leftarrow x}}) + C \cdot \gamma(\sigma_{u \leftarrow x}))) + C \\ &= \lambda(\mathcal{T}_\sigma) + C \cdot \gamma(\sigma) - C(\gamma(t_{\text{var}}(\sigma))) + C \\ &\leq \lambda(\mathcal{T}_\sigma) + C \cdot \gamma(\sigma), \end{aligned}$$

where the equation is by Proposition 6.8, and  $\gamma(t_{\text{var}}(\sigma)) = t_{\text{var}}(\sigma)(0, 0) \geq 1$  is the constant term in  $t_{\text{var}}(\sigma)$  that represents the computation cost for NextVar( $\sigma$ ).  $\square$

For  $(\Phi, \sigma, v)$  satisfying Condition 3.3,  $(\Phi, \sigma_{v \leftarrow \star}, v)$  satisfies Condition 3.8, hence

$$\bar{t}_{\text{MS}}(\Phi, \sigma, v) = (1 - q_v \cdot \theta_v) \bar{t}_{\text{MO}}(\Phi, \sigma_{v \leftarrow \star}, v) + O(1) \leq (1 - q_v \cdot \theta_v) (\lambda(\mathcal{T}_{\sigma_{v \leftarrow \star}}) + O(\gamma(\sigma_{v \leftarrow \star}))) + O(1),$$

where the inequality holds by Lemma 6.9. This proves Lemma 6.7.

**6.3. A random path simulating RCT.** Given a partial assignment  $\sigma \in \mathcal{Q}^*$  and a variable  $v \in V \setminus \Lambda(\sigma)$ , define

$$(18) \quad \begin{aligned} \psi_v^\sigma(\star) &= \frac{1 - q_v \cdot \theta_v}{2 - q_v \cdot \theta_v}, \\ \forall x \in Q_v, \quad \psi_v^\sigma(x) &= \frac{\mu_v^\sigma(x)}{2 - q_v \cdot \theta_v}. \end{aligned}$$

Obviously,  $\psi_v^\sigma(\cdot)$  is a well-defined probability distribution over  $\mathcal{Q}_v^\star$ . The recursive cost tree defined in Definition 6.6 inspires the following random process of partial assignments.

**Definition 6.10** (the Path( $\sigma$ ) process). For any  $\sigma \in \mathcal{Q}^*$ , Path( $\sigma$ ) =  $(\sigma_0, \sigma_1, \dots, \sigma_\ell)$  is a random sequence of partial assignments generated from the initial  $\sigma_0 = \sigma$  as follows: for  $i = 0, 1, \dots$ ,

- (1) if NextVar( $\sigma_i$ ) =  $\perp$ , the sequence stops at  $\sigma_i$ ;
- (2) otherwise  $u = \text{NextVar}(\sigma_i) \in V$ , the partial assignment  $\sigma_{i+1} \in \mathcal{Q}^*$  is generated from  $\sigma_i$  by randomly giving  $\sigma(u)$  a value  $x \in \mathcal{Q}_u^\star$ , such that

$$\forall x \in \mathcal{Q}_u^\star, \quad \Pr[\sigma_{i+1} = (\sigma_i)_{u \leftarrow x}] = \psi_u^\sigma(x).$$

The length  $\ell(\sigma)$  of Path( $\sigma$ ) =  $(\sigma_0, \sigma_1, \dots, \sigma_{\ell(\sigma)})$  is a random variable whose distribution is determined by  $\sigma$ . We simply write  $\ell = \ell(\sigma)$  and Path( $\sigma$ ) =  $(\sigma_0, \sigma_1, \dots, \sigma_\ell)$  if  $\sigma$  is clear from the context.

It is quite obvious that Path( $\sigma$ ) satisfies the Markov property. In fact, Path( $\sigma$ ) can be seen as a Markov chain on space  $\mathcal{Q}^*$  such that any  $\sigma$  with NextVar( $\sigma$ ) =  $\perp$  has a self-loop with probability 1.

For any two partial assignments  $\tau_1, \tau_2 \in \mathcal{Q}^*$ , define

$$(19) \quad \chi(\tau_1, \tau_2) \triangleq \prod_{v \in \Lambda^+(\tau_1) \setminus \Lambda^+(\tau_2)} (2 - q_v \cdot \theta_v),$$

where  $\chi(\tau_1, \tau_2) = 1$  for the case when  $\Lambda^+(\tau_1) \setminus \Lambda^+(\tau_2) = \emptyset$  by convention. With a bit abuse of notation, given any partial assignment  $\sigma \in \mathcal{Q}^*$ , let Path( $\sigma$ ) =  $(\sigma_0, \sigma_1, \dots, \sigma_\ell)$ , we use  $\chi(\sigma)$  to denote  $\chi(\sigma_\ell, \sigma)$ .

The significance of the random process  $\text{Path}(\sigma)$  and the function  $\chi(\cdot, \cdot)$  is that they are related to the complexity of  $\text{MarginSample}$  through the following function: for any sequence  $P = (\sigma_0, \sigma_1, \dots, \sigma_\ell) \in (\mathcal{Q}^*)^{\ell+1}$  with  $\ell \geq 0$ ,

$$(20) \quad H(P) \triangleq \sum_{i=0}^{\ell} (\chi(\sigma_i, \sigma_0) \cdot t_{\text{var}}(\sigma_i)) + \chi(\sigma_\ell, \sigma_0) \cdot \bar{t}_{\text{BF}}(\sigma_\ell),$$

where  $t_{\text{var}}(\cdot)$  and  $\bar{t}_{\text{BF}}(\cdot)$  are defined in Definition 6.5, expressed in form of (15).

Recall the  $\lambda(\mathcal{T}_\sigma)$  defined in (16). We have the following lemma.

**Lemma 6.11.** *Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment. Then*

$$\mathbb{E} [H(\text{Path}(\sigma))] \geq \lambda(\mathcal{T}_\sigma).$$

The following corollary follows immediately by combining Lemma 6.7 and Lemma 6.11.

**Corollary 6.12.** *For any  $(\Phi, \sigma, v)$  satisfying Condition 3.3, let  $\sigma^* = \sigma_{v \leftarrow \star}$ , it holds that*

$$\bar{t}_{\text{MS}}(\Phi, \sigma, v) \leq (1 - q_v \cdot \theta_v) (\mathbb{E} [H(\text{Path}(\sigma^*))]) + O(\mathbb{E} [H(\text{Path}(\sigma^*))]) (0, 0) + O(1).$$

where  $\mathbb{E} [H(\text{Path}(\sigma^*))] (0, 0)$  is the constant term in  $\mathbb{E} [H(\text{Path}(\sigma^*))]$  in the form of (15).

*Proof of Lemma 6.11.* We prove this lemma by an induction on the structure of RCT. The base case is when  $T_\sigma$  is just a single root, in which case  $\text{NextVar}(\sigma) = \perp$ , and by Definition 6.6,

$$\lambda(\mathcal{T}_\sigma) = \rho_\sigma(\sigma) \cdot t_{\text{var}}(\sigma) + \rho_\sigma(\sigma) \cdot \bar{t}_{\text{BF}}(\sigma) = t_{\text{var}}(\sigma) + \bar{t}_{\text{BF}}(\sigma).$$

Also, by  $\text{NextVar}(\sigma) = \perp$  and Definition 6.10 we have  $\text{Path}(\sigma) = (\sigma)$ , and  $\chi(\sigma_\ell, \sigma_0) = 1$ . Hence by (20),

$$\mathbb{E} [H(\text{Path}(\sigma))] = t_{\text{var}}(\sigma) + \bar{t}_{\text{BF}}(\sigma) = \lambda(\mathcal{T}_\sigma),$$

The base case is proved.

For the induction step, we assume that  $T_\sigma$  is a tree of depth  $> 0$ . Thus by Definition 6.6,  $\text{NextVar}(\sigma) = u \neq \perp$  for some  $u \in V$  and  $\ell(\sigma) \geq 1$ . According to Item 2 of Definition 6.10, we have

$$(21) \quad \forall x \in \mathcal{Q}_u^*, \quad \Pr [\sigma_1 = \sigma_{u \leftarrow x}] = \psi_u^\sigma(x).$$

Moreover, by the Markov property, given  $\sigma_1 = \sigma_{u \leftarrow x}$  for each  $x \in \mathcal{Q}_u^*$ , the subsequence  $(\sigma_1, \sigma_2, \dots, \sigma_{\ell(\sigma)})$  is identically distributed as  $\text{Path}(\sigma_{u \leftarrow x})$ . In addition, it can be verified that for any sequence of partial assignments  $P = (\tau_0, \tau_1, \dots, \tau_\ell)$  with  $\ell \geq 1$  satisfying  $\Pr [\text{Path}(\sigma) = P] > 0$ ,

$$(22) \quad H(P) = (2 - q_u \cdot \theta_u) H((\tau_1, \dots, \tau_\ell)).$$

Therefore, conditioning on  $\sigma_1 = \sigma_{u \leftarrow x}$  for each  $x \in \mathcal{Q}_u^*$ , we have

$$(23) \quad \begin{aligned} \mathbb{E} [H(\text{Path}(\sigma)) \mid \sigma_1 = \sigma_{u \leftarrow x}] &= t_{\text{var}}(\sigma) + (2 - q_u \cdot \theta_u) \mathbb{E} [H((\sigma_1, \dots, \sigma_\ell)) \mid \sigma_1 = \sigma_{u \leftarrow x}] \\ &= t_{\text{var}}(\sigma) + (2 - q_u \cdot \theta_u) \mathbb{E} [H(\text{Path}(\sigma_{u \leftarrow x}))]. \end{aligned}$$

Therefore by the law of total expectation, we have

$$(24) \quad \begin{aligned} \mathbb{E} [H(\text{Path}(\sigma))] &= \Pr [\sigma_1 = \sigma_{u \leftarrow \star}] \cdot \mathbb{E} [H(\text{Path}(\sigma)) \mid \sigma_1 = \sigma_{u \leftarrow \star}] \\ &\quad + \sum_{x \in \mathcal{Q}_u} (\Pr [\sigma_1 = \sigma_{u \leftarrow x}] \cdot \mathbb{E} [H(\text{Path}(\sigma)) \mid \sigma_1 = \sigma_{u \leftarrow x}]) \\ &= \frac{1 - q_u \cdot \theta_u}{2 - q_u \cdot \theta_u} \cdot (t_{\text{var}}(\sigma) + (2 - q_u \cdot \theta_u) \mathbb{E} [H(\text{Path}(\sigma_{u \leftarrow \star}))]) \\ &\quad + \sum_{x \in \mathcal{Q}_u} \left( \frac{\mu_u^\sigma(x)}{2 - q_u \cdot \theta_u} \cdot (t_{\text{var}}(\sigma) + (2 - q_u \cdot \theta_u) \mathbb{E} [H(\text{Path}(\sigma_{u \leftarrow x}))]) \right) \\ &= t_{\text{var}}(\sigma) + (1 - q_u \cdot \theta_u) \mathbb{E} [H(\text{Path}(\sigma_{u \leftarrow \star}))] + \sum_{x \in \mathcal{Q}_u} (\mu_u^\sigma(x) \cdot \mathbb{E} [H(\text{Path}(\sigma_{u \leftarrow x}))]) \end{aligned}$$

where the second equality is by (21) and (23), and the last equality is by

$$\frac{1 - q_u \cdot \theta_u}{2 - q_u \cdot \theta_u} + \sum_{x \in Q_u} \frac{\mu_u^\sigma(x)}{2 - q_u \cdot \theta_u} = 1.$$

Note that by Item 2b in Definition 6.6, for each  $x \in Q_u^*$ , the subtree in  $T_\sigma$  rooted by  $\sigma_{u \leftarrow x}$  is precisely the  $T_\tau$  in the RCT  $\mathcal{T}_\tau = (T_\tau, \rho_\tau)$  rooted at  $\tau = \sigma_{u \leftarrow x}$ . By the induction hypothesis,

$$\mathbb{E} [H(\text{Path}(\sigma_{u \leftarrow x}))] \geq \lambda(\mathcal{T}_{\sigma_{u \leftarrow x}}).$$

Combining with (24), we have

$$\mathbb{E} [H(\text{Path}(\sigma))] \geq t_{\text{var}}(\sigma_0) + (1 - q_u \cdot \theta_u) \cdot \lambda(\mathcal{T}_{\sigma_{u \leftarrow \star}}) + \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot \lambda(\mathcal{T}_{\sigma_{u \leftarrow x}})) = \lambda(\mathcal{T}_\sigma).$$

where the equality is by Proposition 6.8.  $\square$

**6.4. Refutation of bad path.** The partial assignment maintained in Algorithm 1 evolves as a random sequence  $X^0, X^1, \dots, X^n$  which was formally defined in Definition 5.6. The efficiency of the Margin-Sample (Algorithm 3) called within in Algorithm 1 crucially relies on that its input partial assignments are generated as this random sequence.

We define a procedure  $\text{Simulate}(\cdot)$  such that  $\text{Simulate}(t)$  generates the prefix  $(X^0, X^1, \dots, X^t)$  of the random partial assignments  $X^0, X^1, \dots, X^n$  maintained in Algorithm 1 defined in Definition 5.6. This is explicitly described in Algorithm 5, which is defined just to facilitate the analysis.

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**Algorithm 5:**  $\text{Simulate}(1 \leq t \leq n)$

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```

1  $X^0 \leftarrow \star^V$ ;
2 for  $i = 1$  to  $t$  do
3   if  $v_i$  is not  $X^{i-1}$ -fixed then
4     choose  $r \in [0, 1)$  uniformly at random;
5     identify the unique  $b \in Q_{v_i}$  satisfying  $\sum_{a < b} \mu_{v_i}^{X^{i-1}}(a) \leq r < \sum_{a \leq b} \mu_{v_i}^{X^{i-1}}(a)$ ;
6      $X^i \leftarrow X_{v_i \leftarrow b}^{i-1}$ ;
7   else
8      $X^i \leftarrow X^{i-1}$ ;
9 return  $(X^0, X^1, \dots, X^t)$ ;
```

---

We may consider  $\text{Simulate}(t-1) = (X^0, X^1, \dots, X^{t-1})$ , and  $X_0^t$  constructed as  $X_0^t = X_{v_t \leftarrow \star}^{t-1}$  with probability  $1 - q_{v_t} \theta_{v_t}$  if  $v_t$  is not  $X^{t-1}$ -fixed, which simulates what is passed to  $\text{MarginOverflow}$  and generates  $\text{Path}(X_0^t) = (X_0^t, X_1^t, \dots, X_\ell^t)$ .

Formally, for  $1 \leq t \leq n$ , we define the following random process:

$$(25) \quad \begin{aligned} (X^0, X^1, \dots, X^{t-1}) &\leftarrow \text{Simulate}(t-1), \\ X_0^t &\leftarrow \begin{cases} X_{v_t \leftarrow \star}^{t-1} & \text{if } v_t \text{ is not } X^{t-1}\text{-fixed and } r_t = 1, \\ X^{t-1} & \text{otherwise,} \end{cases} \\ (X_0^t, X_1^t, \dots, X_\ell^t) &\leftarrow \text{Path}(X_0^t). \end{aligned}$$

where  $r_t$  is sampled from  $\text{Bern}(1 - q_{v_t} \cdot \theta_{v_t})$  independently.

To bound the expected cost of  $\text{MarginSample}$ , we will bound  $\mathbb{E} [H(\text{Path}(X_0^t))]$  where  $H(\cdot)$  is defined in (20). But first, we give a witness for a certain kind of “bad” paths.

Recall the  $\mathcal{C}_{\text{frozen}}^\sigma$  defined in Definition 3.1 and the  $\mathcal{C}_{\star\text{-con}}^\sigma$  in Definition 3.6. Given any  $U \subseteq V$  and  $E \subseteq \mathcal{C}$ , we use  $U \uplus E$  to denote the disjoint union  $U \cup E$ .

**Definition 6.13** ( $\sigma$ -bad variables, constraints and events). Let  $t \in [n]$ ,  $U \subseteq V$ ,  $E \subseteq \mathcal{C}$ ,  $\sigma \in \mathcal{Q}^*$  be a partial assignment, and  $T = U \uplus E$  be a subset of variables and constraints.

- Define  $\mathcal{C}_{\star\text{-frozen}}^\sigma \triangleq \mathcal{C}_{\text{frozen}}^\sigma \cap \mathcal{C}_{\star\text{-con}}^\sigma$ .

- Define  $V_\star^\sigma \triangleq \{v \in V \mid \sigma(v) = \star\}$  to be the set of variables assigned as  $\star$ .
- Let  $\mathcal{E}_T^\sigma$  be the event  $(U = V_\star^{\sigma_\ell}) \wedge (E \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell})$  where  $\sigma_\ell$  is the last partial assignment in the sequence  $\text{Path}(\sigma)$ .
- Let  $\mathcal{E}_T^t$  be the event  $(U = V_\star^{X_\ell^t}) \wedge (E \subseteq \mathcal{C}_{\star\text{-frozen}}^{X_\ell^t})$  where  $(X^0, X^1, \dots, X^{t-1}, X_0^t, X_1^t, \dots, X_\ell^t)$  is constructed as in (25).

Intuitively,  $V_\star^\sigma$ ,  $\mathcal{C}_{\star\text{-frozen}}^\sigma$ ,  $\mathcal{E}_T^\sigma$  and  $\mathcal{E}_T^t$  provide witnesses for the deep recursion of `MarginOverflow`, such that any “bad”  $\text{Path}(\sigma)$  that causes the inefficiency of `MarginOverflow` also creates many “bad” variables in  $V_\star^\sigma$  and constraints in  $\mathcal{C}_{\star\text{-frozen}}^\sigma$ , thus the events  $\mathcal{E}_T^\sigma$  and  $\mathcal{E}_T^t$  happen for some large enough  $T$ .

We first present some basic properties along the  $\text{Path}(\sigma)$ , including the monotonicity of several variable/constraint attributes, and the relation between the length of  $\text{Path}(\sigma)$  and the sizes of  $\mathcal{C}_v^{\sigma_\ell}$ ,  $\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$ ,  $V_\star^{\sigma_\ell}$  and  $\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}$ . Recall that for each  $v \in V^\sigma$ ,  $H_v^\sigma = (V_v^\sigma, \mathcal{C}_v^\sigma)$  defined in Section 3.2 denotes the connected component in  $H^\sigma$  which contains the vertex/variable  $v$ .

**Lemma 6.14.** *Let  $\sigma \in \mathcal{Q}^*$  and  $\text{Path}(\sigma) = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ . For every  $0 \leq i \leq j \leq \ell$ , it holds that*

$$\text{(monotonicity property)} \quad V_\star^{\sigma_i} \subseteq V_\star^{\sigma_j}, \quad \mathcal{C}_P^{\sigma_i} \subseteq \mathcal{C}_P^{\sigma_j},$$

where  $P$  can be any attribute  $P \in \{\text{frozen}, \star\text{-con}, \star\text{-frozen}\}$ .

Moreover, if there is exactly one variable  $v \in V$  having  $\sigma(v) = \star$ , it holds that

$$\text{(upper bound on } |\mathcal{C}_v^{\sigma_\ell}| \text{ and } |\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}|) \quad |\mathcal{C}_v^{\sigma_\ell}| \leq |\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}| \leq \Delta \cdot (|V_\star^{\sigma_\ell}| + |\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}|),$$

$$\text{(upper bound on length of } \text{Path}(\sigma)) \quad \ell \leq k\Delta \cdot (|V_\star^{\sigma_\ell}| + |\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}|).$$

The proof of Lemma 6.14 is through a careful verification of definitions, and is deferred to Appendix B.

We now state two technical lemmas that are crucial for the analysis of the efficiency of `MarginSample` and `RejectionSampling`. These two lemmas essentially provide tail bounds for the witnesses of “bad” paths, which state that a large witness of the “bad” path generated as in (25) occurs with exponentially small probability. We first state the tail bound used in the analysis of `MarginSample`.

**Lemma 6.15.** *Assume  $8\text{ep}\Delta^3 \leq 0.99\alpha$ , where  $\alpha$  is defined as in (6). Let  $1 \leq t \leq n$ . Let  $(X^0, X^1, \dots, X^{t-1}, X_0^t, X_1^t, \dots, X_\ell^t)$  be generated as in (25). For any integer  $i \geq 0$ ,*

$$\Pr \left[ |V_\star^{X_\ell^t}| + |\mathcal{C}_{\star\text{-frozen}}^{X_\ell^t}| \geq i\Delta \right] \cdot \mathbb{E} \left[ \chi(X_0^t) \mid |V_\star^{X_\ell^t}| + |\mathcal{C}_{\star\text{-frozen}}^{X_\ell^t}| \geq i\Delta \right] \leq 2^{-i}.$$

The following tail bound is used in the analysis of `RejectionSampling`.

**Lemma 6.16.** *Assume  $8\text{ep}\Delta^3 \leq 0.99\alpha$ . Let  $(X^0, X^1, \dots, X^n) = \text{Simulate}(n)$ . For any integer  $i \geq 0$ ,*

$$\Pr \left[ |\mathcal{C}_v^{X^n}| \geq 2i\Delta^2 \right] \leq 8ek \cdot 4^{-i}.$$

Lemmas 6.15 and 6.16 are proved through similar arguments, which consist of the following two steps:

- (1) showing an exponential tail bound over the occurrences of “bad” variables and *disjoint* “bad” constraints, which is done by careful analyses of `Simulate` and `Path`,
- (2) boosting the above basic tail bound to the form required as in Lemma 6.15, which is done by using a newly invented combinatorial structure named generalized  $\{2, 3\}$ -tree.

The formal proofs of Lemma 6.15 and Lemma 6.16 are technically involved and are deferred to Section 7.

**6.5. Efficiency of `MarginSample`.** We now prove the following upper bound on the expected running time of `MarginSample` (Algorithm 3), which is expressed in the form of (15).

Let  $T_{\text{MS}}(\Phi, \sigma, v)$  be the random variable that represents the complexity of `MarginSample`( $\Phi, \sigma, v$ ) when Condition 3.3 is satisfied by  $(\Phi, \sigma, v)$ . Note that  $\bar{t}_{\text{MS}}(\Phi, \sigma, v) = \mathbb{E}[T_{\text{MS}}(\Phi, \sigma, v)]$  by Definition 6.4.

**Theorem 6.17.** *Assume  $8\text{ep}\Delta^3 \leq 0.99\alpha$ . Let  $X^0, X^1, \dots, X^n$  be the random sequence in Definition 5.6. Assume the convention that  $T_{\text{MS}}(\Phi, X^{t-1}, v_t) = 0$  when  $v_t$  is  $X^{t-1}$ -fixed. For any  $1 \leq t \leq n$ ,*

$$\mathbb{E} [T_{\text{MS}}(\Phi, X^{t-1}, v_t)] \leq O\left(q^2 k^2 \Delta^9\right) \cdot \underline{x} + 24k\Delta^7 \cdot \underline{y} + O\left(q^3 k^3 \Delta^9\right),$$

where expectation is taken over both  $X^{t-1}$  and the randomness of `MarginSample` algorithm.

The convention is safe to apply since  $\text{MarginSample}(\Phi, \sigma, v)$  is never called when  $v$  is not  $\sigma$ -fixed.

To prove this theorem, one need to give concrete bounds on  $t_{\text{var}}(\sigma)$  and  $\bar{t}_{\text{BF}}(\sigma)$  (Definition 6.5), respectively for the two nontrivial steps in the algorithm: the  $\text{NextVar}(\sigma)$  called at Line 1 and the Bernoulli factory called at Line 10, both in Algorithm 4.

6.5.1. *Cost of NextVar* ( $\sigma$ ). We give an explicit bound on the complexity  $t_{\text{var}}(\sigma)$  (Definition 6.5) for computing the  $\text{NextVar}(\sigma)$  in Line 1 of Algorithm 4, in terms of the size of  $\mathcal{C}_{\star\text{-con}}^\sigma$  (Definition 3.6).

Assume the input model and data structures in Section 6.1. We have the following result.

**Proposition 6.18.** *For any  $\sigma \in \mathcal{Q}^*$ ,  $\text{NextVar}(\sigma)$  can be computed using at most  $|\mathcal{C}_{\star\text{-con}}^\sigma|$  queries to  $\text{Eval}(\cdot)$ ,  $\Delta |\mathcal{C}_{\star\text{-con}}^\sigma|$  queries to  $\text{Frozen}(\cdot)$ , and  $O\left(k^2 \Delta |\mathcal{C}_{\star\text{-con}}^\sigma|^2 + 1\right)$  computation cost, that is*

$$t_{\text{var}}(\sigma) \leq |\mathcal{C}_{\star\text{-con}}^\sigma| \cdot \underline{x} + \Delta |\mathcal{C}_{\star\text{-con}}^\sigma| \cdot \underline{y} + O\left(k^2 \Delta |\mathcal{C}_{\star\text{-con}}^\sigma|^2 + 1\right).$$

As assumed in Section 6.1, a linked list is kept alongside with the partial assignment  $\sigma$  for storing the variables set as  $\star$  in  $\sigma$ . Recall that in Definition 6.13 we use  $V_\star^\sigma$  to denote such set of variables. Recall the simplification  $\Phi^\sigma = (V^\sigma, \mathcal{Q}^\sigma, \mathcal{C}^\sigma)$  of  $\Phi$  under  $\sigma$  and its corresponding hypergraph representation  $H^\sigma = H_{\Phi^\sigma} = (V^\sigma, \mathcal{C}^\sigma)$ , which is formally defined in Section 3.2 and used in the definition of  $\text{NextVar}(\sigma)$ .

The procedure for computing  $\text{NextVar}(\sigma)$  is straightforward on the hypergraph  $H^\sigma$ :

- Perform a depth-first search starting from  $V_\star^\sigma$  on the sub-hypergraph of  $H^\sigma$  induced by  $V^\sigma \cap V_{\text{fix}}^\sigma$  to find the connected components  $V_{\star\text{-con}}^\sigma \supseteq V_\star^\sigma$ .
- Construct the vertex boundary of  $V_{\star\text{-con}}^\sigma$  in  $H^\sigma$ . Return the first boundary vertex  $v_i$  with smallest  $i$  if such vertex exists, and return  $\perp$  if otherwise.

We then verify that this procedure can be implemented within the complexity in Proposition 6.18.

First, observe that the complexity for testing whether a  $c \in \mathcal{C}$  belongs to the sub-hypergraph of  $H^\sigma$  induced by  $V^\sigma \cap V_{\text{fix}}^\sigma$ , is bounded by  $\underline{x} + \Delta \cdot \underline{y} + O(k\Delta)$ , i.e. one query to  $\text{Eval}(\cdot)$ ,  $\Delta$  queries to  $\text{Frozen}(\cdot)$ , and  $O(k\Delta)$  computation cost. This is because it is equivalent to check whether  $c$  is satisfied by  $\sigma$  and  $\text{vbl}(c) \subseteq V_{\text{fix}}^\sigma$ : the former takes one query to  $\text{Eval}(\cdot)$ ; and the latter can be resolved by enumerating all  $c' \in \mathcal{C}$  such that  $\text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset$  and retrieving  $\text{vbl}(c')$  (which costs  $O(k\Delta)$  in computation), and checking whether  $c'$  is  $\sigma$ -frozen for all such  $c'$  (which takes  $\leq \Delta$  queries to  $\text{Frozen}(\cdot)$  in total).

It is not difficult to verify that in above depth-first search, the set of constraints that need to be checked whether belong to the induced sub-hypergraph is in fact just the set  $\mathcal{C}_{\star\text{-con}}^\sigma$ . Then the complexity contributed by testing the membership of constraints in the induced sub-hypergraph is bounded by  $|\mathcal{C}_{\star\text{-con}}^\sigma| \cdot \underline{x} + \Delta |\mathcal{C}_{\star\text{-con}}^\sigma| \cdot \underline{y} + O(k\Delta |\mathcal{C}_{\star\text{-con}}^\sigma|)$ . And this is the only part that may access the oracles, so we have the respective bounds on the queries to the oracles  $\text{Eval}(\cdot)$  and  $\text{Frozen}(\cdot)$ .

For other computation costs, in the depth-first search, the sets of variables and constraints that have been visited can be straightforwardly stored using two dynamic arrays, one for variables and the other for constraints. Querying if some variable/constraint has been visited or updating their status, is done by iterating over the entire array, which takes linear time in the current size of the dynamic array each time a query or an update is conducted. Note that the number of visited constraints is at most  $|\mathcal{C}_{\star\text{-con}}^\sigma|$  and hence the number of visited variables is at most  $k |\mathcal{C}_{\star\text{-con}}^\sigma|$ . Therefore this part costs  $O(k^2 |\mathcal{C}_{\star\text{-con}}^\sigma|^2)$  in computation in total. Overall, the computation cost is easily dominated by  $O(k^2 \Delta |\mathcal{C}_{\star\text{-con}}^\sigma|^2 + 1)$ , where the additional  $O(1)$  is meant to deal with the degenerate case of  $|\mathcal{C}_{\star\text{-con}}^\sigma| = 0$ .

6.5.2. *Cost of Bernoulli factory*. Here we state a complexity bound for the Bernoulli factory in Line 10 of Algorithm 4 that is useful in our analysis of  $\text{MarginSample}$ .

We have the following bound on the  $\bar{t}_{\text{BF}}(\sigma)$  (Definition 6.5) for the Bernoulli factory.

**Proposition 6.19.** *There exist constants  $C_0, C_1 > 0$  such that the following holds. Let  $1 \leq t \leq n$  and let  $\text{Path}(X_0^t) = (X_0^t, X_1^t, \dots, X_\ell^t)$  be generated as in (25), then*

$$\bar{t}_{\text{BF}}(X_\ell^t) \leq C_1 q^2 k^2 \Delta^6 \left( |\mathcal{C}_{\star\text{-con}}^{X_\ell^t}| + 1 \right) (1 - \epsilon a q)^{-|\mathcal{C}_{\star\text{-con}}^{X_\ell^t}|} \cdot \underline{x} + C_0 q^3 k^3 \Delta^6 \left( |\mathcal{C}_{\star\text{-con}}^{X_\ell^t}| + 1 \right) (1 - \epsilon a q)^{-|\mathcal{C}_{\star\text{-con}}^{X_\ell^t}|}.$$

Note that  $X_\ell^t$  is a random variable and the bound in Proposition 6.19 holds for any possible  $X_\ell^t$ .

Next, we prove Proposition 6.19. In Theorem A.4, the following complexity bound is proved for all such  $\sigma \in \mathcal{Q}^*$  where Condition 3.8 is satisfied by  $(\Phi, \sigma, v)$  for some  $v \in V$ :

$$(26) \quad \bar{t}_{\text{BF}}(\sigma) = O\left(q^2 k^2 \Delta^6 (|\mathcal{C}_v^\sigma| + 1)(1 - e\alpha q)^{-|\mathcal{C}_v^\sigma|} \cdot \underline{x} + q^3 k^3 \Delta^6 (|\mathcal{C}_v^\sigma| + 1)(1 - e\alpha q)^{-|\mathcal{C}_v^\sigma|}\right),$$

where  $\mathcal{C}_v^\sigma$  denotes the set of constraints in the connected component that contain  $v$  in  $\Phi^\sigma$ , the simplification of  $\Phi$  under  $\sigma$ .

*Proof of Proposition 6.19.* By the construction in (25),  $X_0^t$  satisfies one of the two cases:

- (1)  $(\Phi, X_0^t, v_t)$  satisfies Condition 3.8 and  $v_t$  is the only variable with  $X_0^t(v_t) = \star$ ;
- (2)  $X_0^t(u) \in Q_u \cup \{\star\}$  for all  $u \in V$ .

For case (1): By Lemma 6.14, we have  $|\mathcal{C}_v^{X_\ell^t}| \leq |\mathcal{C}_{\star\text{-con}}^{X_\ell^t}|$  because  $X_\ell^t$  is generated by  $\text{Path}(X_0^t) = (X_0^t, X_1^t, \dots, X_\ell^t)$  obeying the Markov property. Moreover, by Lemma 5.9 and the construction of  $\text{Path}$ , it holds for sure that Condition 3.8 is satisfied by  $(\Phi, X_\ell^t, u)$  for some  $u \in V$ . Therefore, the complexity bound in (26) always holds for  $X_\ell^t$ , which combined with the relation  $|\mathcal{C}_v^{X_\ell^t}| \leq |\mathcal{C}_{\star\text{-con}}^{X_\ell^t}|$  that we have just established, proves the case.

For case (2): In this case by the definition of  $\text{Path}$ ,  $\text{Path}(X_0^t) = (X_0^t)$  and Condition 3.8 is no longer satisfied by  $(\Phi, X_\ell^t, v) = (\Phi, X_0^t, v)$  for any  $v$ . Thus,  $\bar{t}_{\text{BF}}(X_\ell^t) = 0$  due to Definition 6.5.  $\square$

6.5.3. *Complexity bound for MarginSample.* It remains to bound the complexity of  $\text{MarginSample}$ . We give an upper bound on the weighted function  $H$  defined in (20) for a random path.

**Lemma 6.20.** *Assume  $8ep\Delta^3 \leq 0.99\alpha$ , where  $\alpha$  is fixed as in (6). Let  $1 \leq t \leq n$  and let  $\text{Path}(X_0^t) = (X_0^t, X_1^t, \dots, X_\ell^t)$  be generated as in (25). There exist constants  $C, C' > 0$  such that*

$$\mathbb{E} [H(\text{Path}(X_0^t))] \leq Cq^2 k^2 \Delta^9 \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + C'q^3 k^3 \Delta^9.$$

*Proof.* According to (20), it is sufficient to prove

$$(27) \quad \mathbb{E} \left[ \sum_{i=0}^{\ell} (\chi(X_i^t, X_0^t) \cdot t_{\text{var}}(X_i^t)) \right] \leq 24k\Delta^5 \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + 120C_1 k^3 \Delta^8$$

and

$$(28) \quad \mathbb{E} [\chi(X_\ell^t, X_0^t) \cdot \bar{t}_{\text{BF}}(X_\ell^t)] \leq 20C_2 k^2 q^2 \Delta^9 \cdot \underline{x} + 20C_3 k^3 q^3 \Delta^9.$$

Therefore, by (20) we have

$$\begin{aligned} \mathbb{E} [H(\text{Path}(X_0^t))] &= \mathbb{E} \left[ \sum_{i=0}^{\ell} (\chi(X_i^t, X_0^t) \cdot t_{\text{var}}(X_i^t)) + \chi(X_\ell^t, X_0^t) \cdot \bar{t}_{\text{BF}}(X_\ell^t) \right] \\ &= \mathbb{E} \left[ \sum_{i=0}^{\ell} (\chi(X_i^t, X_0^t) \cdot t_{\text{var}}(X_i^t)) \right] + \mathbb{E} [\chi(X_\ell^t, X_0^t) \cdot \bar{t}_{\text{BF}}(X_\ell^t)] \\ &\leq 24k\Delta^5 \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + 120C_1 k^3 \Delta^8 + 20C_2 k^2 q^2 \Delta^9 \cdot \underline{x} + 20C_3 k^3 q^3 \Delta^9 \\ &\leq (20C_2 + 24)k^2 q^2 \Delta^9 \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + (120C_1 + 20C_3)k^3 q^3 \Delta^9, \end{aligned}$$

where in the second to the last inequality we apply (27) and (28). The lemma follows by taking  $C = 20C_2 + 24$  and  $C' = 120C_1 + 20C_3$ .

It is then sufficient to prove (27) and (28). We first prove (27). We claim that

$$(29) \quad \left| \mathcal{C}_{\star\text{-con}}^{X_\ell^t} \right| \leq \Delta \cdot \left( \left| V_\star^{X_\ell^t} \right| + \left| \mathcal{C}_{\star\text{-frozen}}^{X_\ell^t} \right| \right).$$

By construction of  $X_0^t$  in (25), either  $(\Phi, X_0^t, v_t)$  satisfies Condition 3.8 and  $v_t$  is the only variable with  $X_0^t(v_t) = \star$ , in which case (29) follows from Lemma 6.14; or  $X_0^t$  satisfies for all  $u \in V$ ,  $X_0^t(u) \in Q_u \cup \{\star\}$ , in which case  $\text{Path}(X_0^t) = (X_0^t)$ ,  $\left| \mathcal{C}_{\star\text{-con}}^{X_\ell^t} \right| = \left| \mathcal{C}_{\star\text{-frozen}}^{X_\ell^t} \right| = \left| V_\star^{X_\ell^t} \right| = 0$ , and (29) holds. This proves the claim.



Let  $L \triangleq \left| V_{\star}^{X_t^t} \right| + \left| C_{\star\text{-frozen}}^{X_t^t} \right|$ . By the monotonicity property in Lemma 6.14,  $C_{\star\text{-con}}^{X_t^t} \subseteq C_{\star\text{-con}}^{X_t^t}$  for every  $0 \leq i \leq \ell$ . Combining with (29) we have

$$(30) \quad \left| C_{\star\text{-con}}^{X_i^t} \right| \leq \left| C_{\star\text{-con}}^{X_t^t} \right| \leq \Delta \left( \left| V_{\star}^{X_t^t} \right| + \left| C_{\star\text{-frozen}}^{X_t^t} \right| \right) \leq \Delta L.$$

Then by Proposition 6.18, there exists a constant  $C_1 > 0$  such that for every  $0 \leq i \leq \ell$ ,

$$\begin{aligned} t_{\text{var}}(X_i^t) &\leq \left| C_{\star\text{-con}}^{X_i^t} \right| \cdot \underline{x} + \Delta \left| C_{\star\text{-con}}^{X_i^t} \right| \cdot \underline{y} + C_1 \cdot k^2 \Delta \left| C_{\star\text{-con}}^{X_i^t} \right|^2 + C_1 \\ &\leq \Delta L \cdot \underline{x} + \Delta^2 L \cdot \underline{y} + C_1 \cdot k^2 \Delta^3 L^2 + C_1. \end{aligned}$$

Note that for each  $0 \leq i \leq \ell$  we have  $\chi(X_i^t, X_0^t) \leq \chi(X_t^t, X_0^t) = \chi(X_0^t)$ . Hence,

$$\mathbb{E} \left[ \sum_{i=0}^{\ell} (\chi(X_i^t, X_0^t) \cdot t_{\text{var}}(X_i^t)) \right] \leq \mathbb{E} \left[ (\ell + 1) \cdot \chi(X_0^t) \cdot (\Delta L \cdot \underline{x} + \Delta^2 L \cdot \underline{y} + C_1 \cdot k^2 \Delta^3 L^2 + C_1) \right].$$

By the upper bound on length of  $\text{Path}(\sigma)$  in Lemma 6.14 and the special case that  $\ell = 0$  when  $X_0^t$  satisfies that  $X_0^t(u) \in Q_u \cup \{\star\}$  for all  $u \in V$ , we have  $\ell \leq kL\Delta$ . Thus, the following can be verified in separate cases  $\ell > 0$ ,  $(\ell = 0) \wedge (L = 0)$ , and  $(\ell = 0) \wedge (L > 0)$ :

$$\begin{aligned} &(\ell + 1) \cdot \chi(X_0^t) \cdot (\Delta L \cdot \underline{x} + \Delta^2 L \cdot \underline{y} + C_1 \cdot k^2 \Delta^3 L^2 + C_1) \\ &\leq 2kL\Delta \cdot \chi(X_0^t) \cdot (\Delta L \cdot \underline{x} + \Delta^2 L \cdot \underline{y} + C_1 \cdot k^2 \Delta^3 L^2 + C_1). \end{aligned}$$

Therefore, we have

$$(31) \quad \begin{aligned} &\mathbb{E} \left[ \sum_{i=0}^{\ell} (\chi(X_i^t, X_0^t) \cdot t_{\text{var}}(X_i^t)) \right] \\ &\leq \mathbb{E} \left[ 2kL\Delta \cdot \chi(X_0^t) \cdot (\Delta L \cdot \underline{x} + \Delta^2 L \cdot \underline{y} + C_1 \cdot k^2 \Delta^3 L^2 + C_1) \right] \\ &\leq \mathbb{E} \left[ \chi(X_0^t) \cdot (L^2 \cdot 2k\Delta^2 \cdot \underline{x} + L^2 \cdot 2k\Delta^3 \cdot \underline{y} + L^3 \cdot 2C_1 k^3 \Delta^4 + L \cdot 2C_1 k\Delta) \right]. \end{aligned}$$

Let  $j \triangleq \lfloor \frac{i}{\Delta} \rfloor$ . Let  $\alpha \in \{1, 2, 3\}$ . We have

$$(32) \quad \begin{aligned} \mathbb{E} [\chi(X_0^t) \cdot L^\alpha] &= \sum_{i \geq 0} (\Pr [L = i] \cdot i^\alpha \cdot \mathbb{E} [\chi(X_0^t) \mid L = i]) \\ &\leq \Delta^\alpha \sum_{i \geq 0} ((j+1)^\alpha \cdot \Pr [L = i] \cdot \mathbb{E} [\chi(X_0^t) \mid L = i]) \\ &\leq \Delta^\alpha \sum_{i \geq 0} ((j+1)^\alpha \cdot \Pr [L \geq j\Delta] \cdot \mathbb{E} [\chi(X_0^t) \mid L \geq j\Delta]) \\ &= \Delta^{\alpha+1} \sum_{j \geq 0} ((j+1)^\alpha \cdot \Pr [L \geq j\Delta] \cdot \mathbb{E} [\chi(X_0^t) \mid L \geq j\Delta]), \end{aligned}$$

where the inequalities are by the non-negativity of  $\chi(\cdot)$ . By Lemma 6.15, we have for every  $i \geq 0$ ,

$$(33) \quad \Pr [L \geq i\Delta] \cdot \mathbb{E} [\chi(X_0^t) \mid L \geq i\Delta] \leq 2^{-i}.$$

Combining with (32), we have

$$\mathbb{E} [\chi(X_0^t) \cdot L^\alpha] \leq \Delta^{\alpha+1} \sum_{i \geq 0} ((i+1)^\alpha \cdot 2^{-i}).$$

Note that  $\sum_{i \geq 0} ((i+1) \cdot 2^{-i}) \leq 4$ ,  $\sum_{i \geq 0} ((i+1)^2 \cdot 2^{-i}) \leq 12$  and  $\sum_{i \geq 0} ((i+1)^3 \cdot 2^{-i}) \leq 52$ . Thus, we have

$$(34) \quad \mathbb{E} [\chi(X_0^t) \cdot L^\alpha] \leq \begin{cases} 4\Delta^2 & \text{if } \alpha = 1, \\ 12\Delta^3 & \text{if } \alpha = 2, \\ 52\Delta^4 & \text{if } \alpha = 3. \end{cases}$$

Combining with (31), (27) is immediate.

Next, we prove (28). By Proposition 6.19 and (30), there exist some constants  $C_2, C_3 > 0$  such that

$$\begin{aligned}\bar{t}_{\text{BF}}(X_t^t) &\leq k^2 q^2 \Delta^6 \left( \left| C_{\star\text{-con}}^{X_t^t} \right| + 1 \right) (1 - e\alpha q)^{-\left| C_{\star\text{-con}}^{X_t^t} \right|} (C_2 \cdot \underline{x} + C_3 k q) \\ &\leq k^2 q^2 \Delta^6 \cdot (\Delta L + 1) (1 - e\alpha q)^{-\Delta L} \cdot (C_2 \cdot \underline{x} + C_3 k q).\end{aligned}$$

We have

$$(35) \quad \begin{aligned}\mathbb{E} [\chi(X_t^t, X_0^t) \cdot \bar{t}_{\text{BF}}(X_t^t)] &\leq \mathbb{E} [\chi(X_0^t) k^2 q^2 \Delta^6 \cdot (\Delta L + 1) (1 - e\alpha q)^{-\Delta L} \cdot (C_2 \cdot \underline{x} + C_3 k q)] \\ &= k^2 q^2 \Delta^7 \cdot \mathbb{E} [\chi(X_0^t) (L + 1) (1 - e\alpha q)^{-\Delta L}] \cdot (C_2 \cdot \underline{x} + C_3 k q).\end{aligned}$$

By (6) we have and  $e\alpha q \leq (4\Delta^2)^{-1}$ . Thus  $(1 - e\alpha q)^{-\Delta^2} \leq 1.3$  for each  $\Delta \geq 2$ . Let  $j \triangleq \lfloor \frac{i}{\Delta} \rfloor$ . We have

$$\begin{aligned}\mathbb{E} [\chi(X_0^t) (L + 1) (1 - e\alpha q)^{-\Delta L}] &= \sum_{i \geq 0} \left( \Pr [L = i] \cdot \mathbb{E} [\chi(X_0^t) \mid L = i] \cdot (i + 1) \cdot (1 - e\alpha q)^{-i\Delta} \right) \\ &\leq \sum_{i \geq 0} \left( (i + 1) \cdot 1.3^{i/\Delta} \cdot \Pr [L = i] \cdot \mathbb{E} [\chi(X_0^t) \mid L = i] \right) \\ &\leq \sum_{i \geq 0} \left( (i + 1) \cdot 1.3^{i/\Delta} \cdot \Pr [L \geq i] \cdot \mathbb{E} [\chi(X_0^t) \mid L \geq i] \right) \\ &\leq 2\Delta \sum_{i \geq 0} \left( (j + 1) \cdot 1.3^j \cdot \Pr [L \geq j\Delta] \cdot \mathbb{E} [\chi(X_0^t) \mid L \geq j\Delta] \right) \\ &\leq 2\Delta^2 \sum_{j \geq 0} \left( (j + 1) \cdot 1.3^j \cdot \Pr [L \geq j\Delta] \cdot \mathbb{E} [\chi(X_0^t) \mid L \geq j\Delta] \right),\end{aligned}$$

where the equality is by the law of total expectation and the inequalities are by the non-negativity of  $\chi(\cdot)$ . Combining with (33), we have

$$\mathbb{E} [\chi(X_0^t) (L + 1) (1 - e\alpha q)^{-\Delta L}] \leq 2\Delta^2 \sum_{i \geq 0} ((i + 1) \cdot 0.65^i) \leq 20\Delta^2.$$

Combining with (35), we have

$$\begin{aligned}\mathbb{E} [\chi(X_t^t, X_0^t) \cdot \bar{t}_{\text{BF}}(X_t^t)] &\leq k^2 q^2 \Delta^7 \cdot 20\Delta^2 \cdot (C_2 \cdot \underline{x} + C_3 k q) \\ &\leq 20C_2 k^2 q^2 \Delta^9 \cdot \underline{x} + 20C_3 k^3 q^3 \Delta^9.\end{aligned}$$

Then (28) holds, which finishes the proof of the lemma.  $\square$

Now we can prove Theorem 6.17, the main theorem of Section 6.5.

*Proof of Theorem 6.17.* Let

$$\begin{aligned}U &\triangleq \{ \sigma \in \mathcal{Q}^* \mid \Pr [X^{t-1} = \sigma] > 0 \}, \\ S &\triangleq \{ \sigma \in \mathcal{Q}^* \mid \Pr [X^{t-1} = \sigma] > 0 \wedge v_t \notin V_{\text{fix}}^\sigma \}.\end{aligned}$$

For any  $\sigma \in S$ , we have  $\Pr [X^{t-1} = \sigma] > 0$  and  $v_t \notin V_{\text{fix}}^\sigma$ , therefore by the construction of  $X^0, X^1, \dots, X^n$  in Definition 5.6, there is a positive probability that  $\text{MarginSample}(\Phi, \sigma, v_t)$  is called within Algorithm 1. Hence by Lemma 5.4, Condition 3.3 is satisfied by  $(\Phi, \sigma, v_t)$  for every  $\sigma \in S$ .

For any  $\sigma \in \mathcal{Q}^*$ , let  $\gamma(\sigma) = \mathbb{E} [H(\text{Path}(\sigma))]$  ( $0, 0$ ) denote the constant term in  $\mathbb{E} [H(\text{Path}(\sigma))]$ . By Corollary 6.12, there exist constants  $C_1, C_2 > 0$  such that for every  $\sigma \in S$ ,

$$(36) \quad \bar{t}_{\text{MS}}(\Phi, \sigma, v_t) \leq (1 - q_{v_t} \cdot \theta_{v_t}) (\mathbb{E} [H(\text{Path}(\sigma_{v_t \leftarrow \star}))]) + C_1 \cdot \gamma(\sigma_{v_t \leftarrow \star}) + C_2.$$

And for every  $\sigma \in U \setminus S$ ,  $v_t$  is  $\sigma$ -fixed, and hence  $\bar{t}_{\text{MS}}(\Phi, \sigma, v_t) = \mathbb{E} [T_{\text{MS}}(\Phi, \sigma, v_t)] = 0$  by convention. On the other hand,  $\mathbb{E} [H(\text{Path}(\sigma))]$  is always nonnegative. Thus the following holds trivially:

$$(37) \quad \bar{t}_{\text{MS}}(\Phi, \sigma, v_t) \leq \mathbb{E} [H(\text{Path}(\sigma))] + C_1 \cdot \gamma(\sigma) + C_2.$$

In addition, given  $\sigma \in U$  and  $X^{t-1} = \sigma$ , by (25) we have if  $v_t$  is  $\sigma$ -fixed, then  $X_0^t = \sigma$ ; otherwise,

$$(38) \quad X_0^t = \begin{cases} \sigma_{v_t \leftarrow \star} & \text{with probability } 1 - q_{v_t} \cdot \theta_{v_t} \\ \sigma & \text{with probability } q_{v_t} \cdot \theta_{v_t} \end{cases}.$$

Thus, if  $v_t$  is  $\sigma$ -fixed, by (37) and  $X_0^t = X^{t-1} = \sigma$ , we have

$$(39) \quad \bar{t}_{\text{MS}}(\Phi, X^{t-1}, v_t) \leq \mathbb{E} [H(\text{Path}(X_0^t))] + C_1 \cdot \gamma(X_0^t) + C_2.$$

If  $v_t$  is not  $\sigma$ -fixed, by (38) and the law of total expectation, we have

$$(1 - q_{v_t} \cdot \theta_{v_t}) (\mathbb{E} [H(\text{Path}(\sigma_{v_t \leftarrow \star}))]) + C_1 \cdot \gamma(\sigma_{v_t \leftarrow \star}) + C_2 \leq \mathbb{E} [H(\text{Path}(X_0^t))] + C_1 \cdot \gamma(X_0^t) + C_2.$$

Combining with (36) and  $X^{t-1} = \sigma$ , we also have (39). In summary, conditioning on any possible  $X^{t-1} = \sigma \in U$ , (39) always holds. Hence by the law of total expectation, we have

$$\mathbb{E} [T_{\text{MS}}(\Phi, X^{t-1}, v_t)] = \mathbb{E} [\bar{t}_{\text{MS}}(\Phi, X^{t-1}, v_t)] \leq \mathbb{E} [H(\text{Path}(X_0^t))] + C_1 \cdot \mathbb{E} [\gamma(X_0^t)] + C_2.$$

By Lemma 6.20, for such  $X_0^t$  constructed as above, there exist constants  $C_3, C_4 > 0$  such that

$$\mathbb{E} [H(\text{Path}(X_0^t))] \leq C_3 q^2 k^2 \Delta^9 \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + C_4 q^3 k^3 \Delta^9,$$

which means that

$$\mathbb{E} [T_{\text{MS}}(\Phi, X^{t-1}, v_t)] \leq C_3 q^2 k^2 \Delta^9 \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + (C_1 + 1)C_4 \cdot q^3 k^3 \Delta^9 + C_2. \quad \square$$

**6.6. Efficiency of RejectionSampling.** Recall the random sequence of partial assignments:

$$X^0, X^1, \dots, X^n,$$

maintained in Algorithm 1, as formally defined in Definition 5.6.

Here, we focus on  $X = X^n$ , the partial assignment obtained after all  $n$  iterations of the **for** loop in Line 2 of Algorithm 1, and passed to the RejectionSampling (Algorithm 2) as input.

Recall the following definitions in Section 3.2. For each  $\sigma \in \mathcal{Q}^*$  and  $v \in V^\sigma$ ,  $H_v^\sigma = (V_v^\sigma, \mathcal{C}_v^\sigma)$  denotes the connected component in  $H^\sigma$  that contains the vertex/variable  $v$ , where  $H^\sigma$  is the hypergraph representation for the CSP formula  $\Phi^\sigma$  obtained from the simplification of  $\Phi$  under  $\sigma$ . We further stipulate that  $H_v^\sigma = (V_v^\sigma, \mathcal{C}_v^\sigma) = (\emptyset, \emptyset)$  is the empty hypergraph when  $v \in \Lambda(\sigma)$  is assigned in  $\sigma$ .

For partial assignment  $\sigma \in \mathcal{Q}^*$ , we let  $T_{\text{Rej}}(\sigma)$  be the random variable that represents the complexity of RejectionSampling( $\Phi, \sigma, V \setminus \Lambda(\sigma)$ ), expressed in form of (15). We prove the following bound on the expectation of  $T_{\text{Rej}}(X)$  on the random partial assignment  $X = X^n$ .

**Theorem 6.21.** *Assume  $8\text{ep}\Delta^3 \leq 0.99\alpha$ , where  $\alpha$  is fixed as in (6).*

$$\mathbb{E} [T_{\text{Rej}}(X)] \leq 128\text{ek}\Delta^4 n \cdot \underline{x} + O(qk\Delta^4 n),$$

where expectation is taken over both  $X$  and the randomness of RejectionSampling algorithm.

*Proof of Theorem 6.21.* Let  $\{H_i^X = (V_i^X, \mathcal{C}_i^X) \mid 1 \leq i \leq K\}$  be the connected components in  $H^X$  constructed in Line 1 Algorithm 2. For each  $1 \leq i \leq K$ , let  $T(H_i^X)$  denote the cost contributed by  $H_i^X = (V_i^X, \mathcal{C}_i^X)$  during the repeat loop in Algorithm 2. Then the total cost is given by

$$(40) \quad T_{\text{Rej}}(X) \leq \Delta n \cdot \underline{x} + O(\Delta n) + \sum_{i \in [K]} T(H_i^X).$$

This is because:

- the cost of Line 1 is at most  $n\Delta \cdot \underline{x} + O(n\Delta)$ , because it uses at most  $|\mathcal{C}| \leq \Delta n$  queries to  $\text{Eval}(\cdot)$ , which contributes the  $\Delta n \cdot \underline{x}$  term, and a depth-first search that visits all variables and constraints to compute  $H_1^X, H_2^X, \dots, H_K^X$ , which costs  $O(\Delta n)$  in computation.
- the cost of Lines 2-5 for each component  $H_i^X$  is  $T(H_i^X)$ .

Alternatively, for each  $v \in V$ , we use  $T(H_v^X)$  to denote the  $T(H_i^X)$  for the  $1 \leq i \leq K$  with  $v \in V_i^X$ ; and let  $T(H_v^X) = 0$  if there is no such  $1 \leq i \leq K$ , which occurs when  $v \in \Lambda(X)$  is assigned in  $X$ . Clearly,

$$\sum_{i \in [K]} T(H_i^X) \leq \sum_{v \in V} T(H_v^X).$$

Combining with (40), we have

$$(41) \quad \mathbb{E} [T_{\text{Rej}}(X)] \leq n\Delta \cdot \underline{x} + O(n\Delta) + \sum_{v \in V} \mathbb{E} [T(H_v^X)].$$

We claim that for each  $v \in V$ ,

$$(42) \quad \mathbb{E} [T(H_v^X) \mid X] \leq |\mathcal{C}_v^X| \cdot (1 - \epsilon\alpha q)^{-|\mathcal{C}_v^X|} \cdot \underline{x} + O\left(kq \cdot (|\mathcal{C}_v^X| + 1) \cdot (1 - \epsilon\alpha q)^{-|\mathcal{C}_v^X|}\right).$$

The degenerate case with  $H_v^X = (\emptyset, \emptyset)$  is trivial. We then assume  $H_v^X \neq (\emptyset, \emptyset)$ . It is well known that the expected number of trials (the iterations of the **repeat** loop in Line 5) taken by the rejection sampling until success is given by  $\mathbb{P}_{H_v^X}[\Omega_{H_v^X}]^{-1}$ , where  $\Omega_{H_v^X}$  denotes the set of satisfying assignments of the CSP of  $H_v^X$  and hence  $\mathbb{P}_{H_v^X}[\Omega_{H_v^X}]$  gives the probability that a uniform random assignment  $\sigma \in \mathcal{Q}_{V_v^X}$  is satisfying for  $\Phi_v^X$ , the CSP formula correspond to the component  $H_v^X$ .

By Theorem 4.1 and Lemma 5.8, we have

$$\mathbb{P}_{H_v^X}[\Omega_{H_v^X}] \geq (1 - \epsilon\alpha q)^{|\mathcal{C}_v^X|}.$$

Therefore, it takes  $(1 - \epsilon\alpha q)^{-|\mathcal{C}_v^X|}$  iterations in expectation to successfully sample the assignment on  $V_v^X$ . And within each iteration, it is easy to verify that at most  $|\mathcal{C}_v^X|$  queries to  $\text{Eval}(\cdot)$  and  $O(k|\mathcal{C}_v^X| + q|V_v^X|) = O(qk(|\mathcal{C}_v^X| + 1))$  computation cost are spent. This proves the claim (42).

We then bound the expectation of (42) over  $X$ . Let  $\ell \triangleq \lfloor t/(2\Delta^2) \rfloor$ . By (6), we have  $\epsilon\alpha q \leq (2\Delta^2)^{-1}$  and then

$$(1 - \epsilon\alpha q)^{-(\ell+1)\Delta^2} \leq 2^{\ell+1}.$$

We have

$$\begin{aligned} \sum_{t \geq 0} \left( (t+1)(1 - \epsilon\alpha q)^{-t} \Pr[|\mathcal{C}_v^X| = t] \right) &\leq 2\Delta^2 \sum_{t \geq 0} \left( \Pr[|\mathcal{C}_v^X| \geq 2\ell\Delta^2] \cdot (\ell+1) \cdot (1 - \epsilon\alpha q)^{-(\ell+1)\Delta^2} \right) \\ &\leq 4\Delta^2 \sum_{t \geq 0} \left( (\ell+1) \cdot 2^\ell \cdot \Pr[|\mathcal{C}_v^X| \geq 2\ell\Delta^2] \right) \\ &= 4\Delta^4 \sum_{\ell \geq 0} \left( (\ell+1) \cdot 2^\ell \cdot \Pr[|\mathcal{C}_v^X| \geq 2\ell\Delta^2] \right) \\ \text{(by Lemma 6.16)} \quad &\leq 32ek\Delta^4 \sum_{\ell \geq 0} \left( (\ell+1) \cdot 2^{-\ell} \right) \\ &\leq 128ek\Delta^4. \end{aligned}$$

Therefore,

$$\mathbb{E} \left[ \left( |\mathcal{C}_v^X| + 1 \right) \cdot (1 - \epsilon\alpha q)^{-|\mathcal{C}_v^X|} \right] = \sum_{t \geq 0} \left( (t+1)(1 - \epsilon\alpha q)^{-t} \Pr[|\mathcal{C}_v^X| = t] \right) \leq 128ek\Delta^4.$$

Thus, there exists a constant  $C > 0$  such that the expectation of (42) is bounded by

$$\mathbb{E} [T(H_v^X)] \leq 128ek\Delta^4 \cdot \underline{x} + 128eCk^2q\Delta^4.$$

The theorem follows by (41).  $\square$

**6.7. Efficiency of the main sampling algorithm.** We now prove Theorem 6.1 and Theorem 6.3.

*Proof of Theorem 6.1.* Assuming the LLL condition in Equation (3), we have  $8\epsilon p\Delta^3 \leq 0.99\alpha$  for  $\alpha$  set as in (6), which is the regime of parameters assumed by Theorem 6.17 and Theorem 6.21.

The followings are the nontrivial costs in Algorithm 1:

- The initialization in Line 1 and the testing of being  $X^{t-1}$ -fixed for  $v_t$  in Line 3 for every  $1 \leq t \leq n$ , which altogether cost:

$$\Delta n \cdot \underline{y} + O(\Delta n),$$

because each  $v_t$  queries  $\text{Frozen}(\cdot)$  for at most  $|c \in \mathcal{C} \mid v_t \in \text{vbl}(c)| \leq \Delta$  times and also costs  $O(\Delta)$  in computation to retrieve all such  $c \in \mathcal{C}$  that  $v_t \in \text{vbl}(c)$ .

- The calls to  $\text{MarginSample}(\Phi, X^{t-1}, v_t)$  at Line 4, which according to Theorem 6.17, costs in total:

$$\sum_{t=1}^n \mathbb{E} [T_{\text{MS}}(\Phi, X^{t-1}, v_t)] \leq O\left(q^2 k^2 \Delta^9 n\right) \cdot \underline{x} + 24k\Delta^6 n \cdot \underline{y} + O\left(q^3 k^3 \Delta^9 n\right).$$

- The final call to  $\text{RejectionSampling}(\Phi, X^n, V \setminus \Lambda(X^n))$  at Line 5, which by Theorem 6.21, costs:

$$\mathbb{E} [T_{\text{Rej}}(X^n)] \leq 128ek\Delta^4 n \cdot \underline{x} + O(qk\Delta^4 n).$$

The overall complexity of Algorithm 1 in expectation is bounded by

$$(43) \quad O(q^2 k^2 \Delta^9 n) \cdot \underline{x} + 24k\Delta^6 n \cdot \underline{y} + O(q^3 k^3 \Delta^9 n).$$

The theorem is proved.  $\square$

Next, we construct the Algorithm 1' for approximate sampling and prove Theorem 6.3.

Given as input a CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  with  $n = |V|$  variables and an error bound  $\varepsilon \in (0, 1)$ , Algorithm 1' does the followings. Set the parameters as:

$$(44) \quad \delta = 0.005 \quad \text{and} \quad N = \left\lceil \frac{\ln(200k\Delta^6 n \varepsilon^{-2})}{0.33\alpha\delta^2} \right\rceil.$$

Algorithm 1' simply executes Algorithm 1 on input  $\Phi$ , with the oracle  $\text{Frozen}(\cdot)$  replaced by the following explicitly implemented Monte Carlo subroutine:

- Given as input any constraint  $c \in \mathcal{C}$  and any partial assignment  $\sigma \in \mathcal{Q}^*$ , repeat for  $N$  times:
  - generate an assignment  $Y \in \mathcal{Q}_{\text{vbl}(c)}$  on  $\text{vbl}(c)$  uniformly at random consistent with  $\sigma$ ;
  - check whether  $c(Y) = \text{True}$  by querying  $\text{Eval}(c, Y)$ ;
- let  $Z$  be the number of times within  $N$  trials that  $c(Y) = \text{False}$ , and return  $I[Z/N > 0.995\alpha]$ .

We further apply a standard memoization trick to guarantee the consistency of the oracle  $\text{Frozen}(\cdot)$  as required in Assumption 2. Each constraint  $c \in \mathcal{C}$  is associated with a deterministic dynamic dictionary  $\text{Dic}_c$  which stores key-value pairs in the form of  $(\tau, \text{ans})$  with  $\tau \in \bigotimes_{v \in \text{vbl}(c)} (\mathcal{Q}_v \cup \{\star, \star\})$  and  $\text{ans} \in \{0, 1\}$ . Upon each query to  $\text{Frozen}(c, \sigma)$ , we first lookup in  $\text{Dic}_c$  for the key  $\sigma_{\text{vbl}(c)}$ . If an  $\text{ans} \in \{0, 1\}$  is retrieved, the query to  $\text{Frozen}(c, \sigma)$  is answered with  $\text{ans}$ ; and if otherwise, an  $\text{ans} = I[Z/N > 0.995\alpha] \in \{0, 1\}$  is computed using the above Monte Carlo subroutine, the key-value pair  $(\sigma_{\text{vbl}(c)}, \text{ans})$  is inserted into  $\text{Dic}_c$  and the query to  $\text{Frozen}(c, \sigma)$  is answered with  $\text{ans}$ . Using a *Trie* data structure for the deterministic dynamic dictionary  $\text{Dic}_c$  incurs an  $O(k)$  computation cost for each query and update. Overall, the resulting algorithm is Algorithm 1'.

*Proof of Theorem 6.3.* In Algorithm 1', each query to the oracle  $\text{Frozen}(\cdot)$  made in Algorithm 1 is implemented using  $N$  queries to the evaluation oracle  $\text{Eval}(\cdot)$  along with  $O(kN)$  computation cost, where  $N$  is set as in (44). By Theorem 6.1, the complexity of Algorithm 1 in expectation is bounded as (43). By replacing

$$\underline{y} \leftarrow N \cdot \underline{x} + O(kN),$$

the expected complexity of Algorithm 1' is bounded by

$$O\left(q^2 k^2 \Delta^9 n \log\left(\frac{\Delta n}{\varepsilon}\right) \cdot \underline{x} + q^3 k^3 \Delta^9 n \log\left(\frac{\Delta n}{\varepsilon}\right)\right).$$

By Chernoff bound, one can verify that for each query to  $\text{Frozen}(c, \sigma)$ , the two cases  $\mathbb{P}[-c \mid \sigma] > \alpha$  and  $\mathbb{P}[-c \mid \sigma] \leq 0.99\alpha$  are distinguished correctly except for an error probability bounded by:

$$2 \exp\left(-\frac{\delta^2}{3} 0.99\alpha N\right) \leq \frac{\varepsilon}{2M} \quad \text{where } M \triangleq 50k\Delta^6 n \varepsilon^{-1}.$$

Due to (43), the expected number of queries to  $\text{Frozen}(\cdot)$  made in Algorithm 1 is at most  $24k\Delta^6 n \leq \frac{\varepsilon M}{2}$ .

By Markov's inequality, the probability that the number of queries to  $\text{Frozen}(\cdot)$  made in Algorithm 1 exceeds  $M$  is at most  $\frac{\varepsilon}{2}$ . By a union bound, with probability at least  $1 - \varepsilon$ , no error occurs in any query to  $\text{Frozen}(\cdot)$ . Then by a coupling between Algorithm 1 and Algorithm 1', the total variation distance between the outputs of the two algorithms on the same input CSP  $\Phi$  can be bounded within  $\varepsilon$ .  $\square$

The following is a formal restatement of Theorem 1.5 and Theorem 1.6. Here the  $\tilde{O}(\cdot)$  hides polylogarithmic factors. The algorithm is just the Monte Carlo method that uses  $\text{MarginSample}(\Phi, \star^V, v)$  (Algorithm 3) as a subroutine for sampling from the marginal distribution  $\mu_v$ .

**Theorem 6.22.** *There are algorithms for marginal sampling and probabilistic inference such that given as input  $\varepsilon, \delta \in (0, 1)$ , CSP formula  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  satisfying (3), and  $v \in V$ , the algorithms perform as follows:*

- *The algorithm for marginal sampling returns a random value  $x \in Q_v$  distributed approximately as  $\mu_v$  within total variation distance  $\varepsilon$ , in expectation using at most  $O(q^2 k^2 \Delta^9 \log(k\Delta/\varepsilon))$  queries to  $\text{Eval}(\cdot)$  and  $O(q^3 k^3 \Delta^9 \log(k\Delta/\varepsilon))$  computation cost.*
- *The algorithm for inference returns a  $\hat{\mu}_v \in [0, 1]^{Q_v}$  using at most  $\tilde{O}(q^3 k^2 \Delta^9 \varepsilon^{-2} \log(1/\delta))$  queries to  $\text{Eval}(\cdot)$  and  $\tilde{O}(q^4 k^3 \Delta^9 \varepsilon^{-2} \log(1/\delta))$  computation cost such that*

$$\Pr[\forall x \in Q_v : (1 - \varepsilon)\mu_v(x) \leq \hat{\mu}_v(x) \leq (1 + \varepsilon)\mu_v(x)] \geq 1 - \delta.$$

*Proof.* When (3) is satisfied, we have  $8\epsilon p \Delta^3 \leq 0.99\alpha$  for the  $\alpha$  set as in (6). Then Condition 3.3 is satisfied by  $(\Phi, \star^V, v)$ . According to Theorem 5.5 and Theorem 6.17,  $\text{MarginSample}(\Phi, \star^V, v)$  returns a perfect sample from  $\mu_v$  with expected cost:

$$O(q^2 k^2 \Delta^9) \cdot \underline{x} + 24k\Delta^6 \cdot \underline{y} + O(q^3 k^3 \Delta^9).$$

Using the same Monte Carlo simulation of  $\text{Frozen}(\cdot)$  as in the proof of Theorem 6.3, but this time with sufficiently large parameter  $N = O(\frac{1}{\alpha} \log(k\Delta/\varepsilon)) = O(q^2 k \Delta^2 \log(k\Delta/\varepsilon))$ , with the substitution  $\underline{y} \leftarrow N \cdot \underline{x} + O(kN)$ , the  $\text{MarginSample}(\Phi, \star^V, v)$  is transformed into an approximate sampler for  $\mu_v$  with bias at most  $\varepsilon/2$  in total variation distance, with the following cost:

$$O(q^2 k^2 \Delta^9 \log(k\Delta/\varepsilon)) \cdot \underline{x} + O(q^3 k^3 \Delta^9 \log(k\Delta/\varepsilon)).$$

This proves the marginal sampler part of the algorithm.

By the local uniformity property stated in Corollary 4.3, assuming (3), we have  $\mu_v(x) > \frac{1}{2q}$ . Thus, by Chernoff bound, each  $\mu_v(x)$  for  $x \in Q_v$  can be estimated within  $(1 \pm \varepsilon)$ -multiplicative precision with probability  $> 0.9$  using  $O(q/\varepsilon^2)$  approximate samples, which in expectation costs in total:

$$(45) \quad \tilde{O}(q^3 k^2 \Delta^9 \varepsilon^{-2}) \cdot \underline{x} + \tilde{O}(q^4 k^3 \Delta^9 \varepsilon^{-2}).$$

By Markov's inequality, the total cost exceeds this bound with probability  $< 0.1$ . By truncation, this gives us an algorithm with fixed cost bounded as (45) so for each  $x \in Q_v$ , with probability  $> 0.8$  the algorithm estimates the value of  $\mu_v(x)$  within  $(1 \pm \varepsilon)$ -multiplicative precision. The algorithm asserted by the theorem is then given by repeating this for  $O(\log \frac{q}{\delta})$  times and applying the median trick.  $\square$

## 7. THE GENERALIZED $\{2, 3\}$ -TREE

In this section, we prove the tail bounds stated in Lemmas 6.15 and 6.16 in Section 6. These two tail bounds basically state that the large witness of long running time occurs with an exponentially small probability, and are both proved with a new combinatorial structure called “generalized  $\{2, 3\}$ -tree”. The generalized  $\{2, 3\}$ -tree is a refinement of  $\{2, 3\}$ -tree, which is previously used in [GLLZ19, JPV21b]. Rather than treated similarly in  $\{2, 3\}$ -tree, the “bad” variables and constraints are treated separately in generalized  $\{2, 3\}$ -tree, because the densities of these two types of bad events are different.

Given a hypergraph  $H = (V, \mathcal{E})$ , let  $\text{Lin}(H)$  be the line graph of  $H$  whose vertex set is the hyperedges in  $\mathcal{E}$  and two hyperedges in  $\mathcal{E}$  are adjacent in  $\text{Lin}(H)$  if and only if they share some vertex in  $H$ . Let  $\text{dist}_{\text{Lin}(H)}(\cdot, \cdot)$  be the shortest path distance in  $\text{Lin}(H)$ .

**Definition 7.1.** (generalized  $\{2, 3\}$ -tree) Given a hypergraph  $H = (V, \mathcal{E})$ , a set  $T = U \cup E$  where  $U \subseteq V$  and  $E \subseteq \mathcal{E}$ , let  $G(T)$  be a directed graph constructed on the set  $T$  such that, for any  $u, v \in T$  there is an arc from  $u$  to  $v$  if and only if at least one of the following conditions is satisfied:

- $u, v \in E$  and  $\text{dist}_{\text{Lin}(H)}(u, v) = 2$  or  $3$ ;
- $u \in U, v \in E$  and there exists  $e \in \mathcal{E}$  such that  $u \in e \wedge \text{dist}_{\text{Lin}(H)}(v, e) = 1$ ;
- $u \in E, v \in U$  and there exists  $e \in \mathcal{E}$  such that  $v \in e \wedge \text{dist}_{\text{Lin}(H)}(u, e) = 1$  or  $2$ ;
- $u, v \in U$  and there exists  $e \in \mathcal{E}$  such that  $u, v \in e$ .

Then  $T$  is a *generalized  $\{2, 3\}$ -tree* of  $H$  if the followings hold:

- (1) for all distinct  $u, v \in E$ ,  $\text{dist}_{\text{Lin}(H)}(u, v) \geq 2$ ;
- (2)  $G(T)$  has a rooted directed spanning tree.

Furthermore, each  $v \in T$  is called a *root* of  $T$  if it is a root of some rooted directed spanning tree of  $G(T)$ . For any  $v \in V$  and intergers  $r, \ell, t \geq 0$ , define

$$\begin{aligned}\mathcal{T}_v^{r,\ell}(H) &\triangleq \{\text{generalized } \{2, 3\}\text{-tree } T \text{ of } H \mid (v \text{ is a root of } T) \wedge (|T \cap V| = r) \wedge (|T \cap \mathcal{E}| = \ell)\}, \\ \mathcal{T}_v^t(H) &\triangleq \{\text{generalized } \{2, 3\}\text{-tree } T \text{ of } H \mid (v \text{ is a root of } T) \wedge (|T \cap V| = r + \Delta \cdot |T \cap \mathcal{E}| = t)\}.\end{aligned}$$

We will use  $\mathcal{T}_v^t$  and  $\mathcal{T}_v^{r,\ell}$  to stand for  $\mathcal{T}_v^t(H)$  and  $\mathcal{T}_v^{r,\ell}(H)$  respectively if  $H$  is clear from the context.

The generalized  $\{2, 3\}$ -tree in Definition 7.1 is inspired by the the notion of  $\{2, 3\}$ -tree defined for the line graph  $\text{Lin}(H)$  [Alo91]. We extend this notion to the original hypergraph  $H$  to simultaneously depict the distances between vertices and hyperedges in  $H$ . We further allow including vertices in  $H$  to some generalized  $\{2, 3\}$ -tree  $T$  granted that all vertices and hyperedges included are close enough to each other. One can verify that every  $\{2, 3\}$ -tree in  $\text{Lin}(H)$  is also a generalized  $\{2, 3\}$ -tree in  $H$ .

Specifically, when the underlying hypergraph in Definition 7.1 is the hypergraph representation  $H_\Phi = (V, \mathcal{C})$  of some CSP  $\Phi$ , a generalized  $\{2, 3\}$ -tree  $T \subseteq V \cup \mathcal{C}$  in  $H_\Phi$  becomes a subset of variables and constraints.

**7.1. Boosting of tail bounds using generalized  $\{2, 3\}$ -trees.** In this section we prove Lemmas 6.15 and 6.16. As stated in Section 6.4, the tail bounds in these two lemmas are proved by boosting two “basic” tail bounds over the occurrences of “bad” variables and *disjoint* “bad” constraints using generalized  $\{2, 3\}$ -tree.

Recall the bad event  $\mathcal{E}_T^t$  in Definition 6.13. The following lemma is used in the proof of Lemma 6.15, which provides a tail bound for the bad event  $\mathcal{E}_T^t$  when the constraints in  $T$  are disjoint. It states that within an LLL regime, the bad event  $\mathcal{E}_T^t$  that may lead to the inefficiency of MarginSample rarely occurs.

**Lemma 7.2.** *Assume  $8\text{ep}\Delta^3 \leq 0.99\alpha$ , where  $\alpha$  is defined as in (6). Let  $1 \leq t \leq n$  and  $(X^0, X^1, \dots, X^{t-1}, X_0^t, X_1^t, \dots, X_t^t)$  be generated as in (25). For any subset of variables and constraints  $T = U \uplus E$  such that the constraints in  $E$  are disjoint, we have*

$$(46) \quad \Pr[\mathcal{E}_T^t] \cdot \mathbb{E}[\chi(X_0^t) \mid \mathcal{E}_T^t] \leq (8ek\Delta)^{-|U|} \cdot (4e\Delta^3)^{-|E|}.$$

The following lemma is used in the proof of Lemma 6.16. It provides another tail bound for a bad event over a set of disjoint constraints that may lead to the inefficiency of RejectionSampling.

**Lemma 7.3.** *Assume  $8\text{ep}\Delta^3 \leq 0.99\alpha$ . Let  $(X^0, X^1, \dots, X^n) = \text{Simulate}(n)$ . For any set of disjoint constraints  $T \subseteq \mathcal{C}$ ,*

$$\Pr[T \subseteq \mathcal{C}_{\text{frozen}}^{X^n}] \leq (4e\Delta^3)^{-|T|}.$$

These are the two basic tail bounds. They will be proved later in Section 7.2. For the rest of Section 7.1, we prove Lemma 6.15 and Lemma 6.16 assuming Lemma 7.2 and Lemma 7.3.

We need the following lemma to bound the sum of weights of the generalized  $\{2, 3\}$ -trees in  $\mathcal{T}_v^t$ .

**Lemma 7.4.** *Let  $H = (V, \mathcal{E})$  be a hypergraph such that each hyperedge contains at most  $k$  vertices and shares vertices with at most  $\Delta$  hyperedges. For any  $v \in V$  and  $t > 0$ , we have*

$$\sum_{T \in \mathcal{T}_v^t} \left( (8ek\Delta)^{-|T \cap V|} (4e\Delta^3)^{-|T \cap \mathcal{E}|} \right) \leq 2^{-\lfloor t/\Delta \rfloor - 1} \cdot \Delta^{-1}.$$

From now on, we denote  $w_1 \triangleq (8ek\Delta)^{-1}$  and  $w_2 \triangleq (4e\Delta^3)^{-1}$ . The following definition is used in the proof of Lemma 7.4, which is inspired by the fact that each hyperedge of  $H$  contains at most  $k$  vertices and shares vertices with at most  $\Delta$  hyperedges.

**Definition 7.5** ( $\{a, b\}$ -labelled tree). An  $\{a, b\}$ -labelled tree is a rooted directed tree where

- each node is labelled  $a$  or  $b$ ;
- each node labelled  $a$  has weight  $w_1x$  and each node labelled  $b$  has weight  $w_2x^\Delta$ .

For each  $\{a, b\}$ -labelled tree  $T$ , let the weight of  $T$  be the product of the weights of all the nodes in  $T$ . With a bit abuse of notation, let  $a(T)$  and  $b(T)$  be the numbers of nodes with label  $a$  and  $b$  in  $T$ , respectively. Given  $z \in \{a, b\}$ , define a tree  $\mathcal{T}_z$  with infinite nodes as follows:

- $\mathcal{T}_z$  is a  $\{a, b\}$ -labelled tree where the root is labelled  $z$ ;
- each node labelled  $a$  has  $k\Delta$  children labelled  $a$  and  $\Delta^2$  children labelled  $b$ ;
- each node labelled  $b$  has  $k\Delta^2$  children labelled  $a$  and  $\Delta^3$  children labelled  $b$ .

A *proper subtree*  $T$  of  $\mathcal{T}_z$  is a rooted directed subtree of  $\mathcal{T}_z$  where the root of  $\mathcal{T}_z$  is also the root of  $T$ . Obviously,  $T$  is also an  $\{a, b\}$ -labelled tree. Define  $\mathcal{T}_z^t \triangleq \{\text{proper subtree } T \text{ of } \mathcal{T}_z \mid a(T) + \Delta \cdot b(T) = t\}$ .

Now we can prove Lemma 7.4.

*Proof of Lemma 7.4.* For each  $v \in V$ , we claim that there exists an injection from the generalized  $\{2, 3\}$ -trees in  $\mathcal{T}_v^t$  to the  $\{a, b\}$ -labelled trees in  $\mathcal{T}_a^t$  such that each  $T \in \mathcal{T}_v^t$  is mapped to some  $T' \in \mathcal{T}_a^t$  satisfying  $|T \cap V| = a(T')$  and  $|T \cap \mathcal{E}| = b(T')$ . Recall that  $w_1 = (8ek\Delta)^{-1}$  and  $w_2 = (4e\Delta^3)^{-1}$ . We have

$$\sum_{T \in \mathcal{T}_v^t} \left( (8ek\Delta)^{-|T \cap V|} (4e\Delta^3)^{-|T \cap \mathcal{E}|} \right) \leq \sum_{T \in \mathcal{T}_a^t} w_1^{a(T)} w_2^{b(T)}.$$

Thus, to prove the lemma, it is sufficient to prove

$$(47) \quad \sum_{T \in \mathcal{T}_a^t} w_1^{a(T)} w_2^{b(T)} \leq 2^{-\lfloor t/\Delta \rfloor - 1} \cdot \Delta^{-1}.$$

Let  $f_1$  be the sum of the weights over all proper subtrees of  $\mathcal{T}_a$ . Similarly, Let  $f_2$  be the sum of the weights over all proper subtrees of  $\mathcal{T}_b$ . By Definition 7.5, one can verify that

$$\begin{aligned} f_1 &= w_1 x (1 + f_1)^{k\Delta} \cdot (1 + f_2)^{\Delta^2}, \\ f_2 &= w_2 x^\Delta (1 + f_1)^{k\Delta^2} \cdot (1 + f_2)^{\Delta^3}. \end{aligned}$$

In addition, let  $[x^m]f_1(x)$  be the coefficient of  $x^m$  in  $f_1$ . By Definition 7.5, we also have

$$(48) \quad \forall m \geq 0, \quad [x^m]f_1(x) = \sum_{T \in \mathcal{T}_a^m} w_1^{a(T)} w_2^{b(T)}.$$

Define

$$h \triangleq x(1 + f_1)^{k\Delta} \cdot (1 + f_2)^{\Delta^2}.$$

We have  $f_2 = w_2 h^\Delta$ ,  $f_1 = w_1 h$  and then

$$h = x(1 + w_1 h)^{k\Delta} \cdot (1 + w_2 h^\Delta)^{\Delta^2}.$$

By applying the Lagrange inversion theorem, we have for each  $m \geq 1$ ,

$$(49) \quad \begin{aligned} [x^m]h(x) &= \frac{1}{m} [u^{m-1}] \left( (1 + w_1 u)^{k\Delta} \cdot (1 + w_2 u^\Delta)^{\Delta^2} \right)^m \\ &= \frac{1}{m} \sum_{i=0}^{\lfloor m/\Delta \rfloor} \left( [u^{m-1-\Delta i}] (1 + w_1 u)^{k\Delta m} \cdot [u^{\Delta i}] (1 + w_2 u^\Delta)^{\Delta^2 m} \right) \end{aligned}$$

Assume  $m \geq 3$  and  $0 \leq i \leq \lfloor \frac{m}{2\Delta} \rfloor$ . Then we have  $m \leq 4(m-1-\Delta i)$ . In addition, for each  $0 < \gamma \leq \beta$  where  $\gamma, \beta$  are integers,

$$(50) \quad \binom{\beta}{\gamma} \leq \left( \frac{e\beta}{\gamma} \right)^\gamma.$$



Thus, we have

$$\begin{aligned}
& [u^{m-1-\Delta i}] (1 + w_1 u)^{k\Delta m} \\
&= \binom{k\Delta m}{m-1-\Delta i} w_1^{m-1-\Delta i} \\
(51) \quad & \text{(by (50))} \leq \left( \frac{ek\Delta m}{m-1-\Delta i} \right)^{m-1-\Delta i} w_1^{m-1-\Delta i} \\
& \text{(by } m \leq 4(m-1-\Delta i) \text{)} \leq (4ek\Delta w_1)^{m-1-\Delta i} \\
& \text{(by } w_1 = (8ek\Delta)^{-1} \text{)} = 2^{1+\Delta i-m}.
\end{aligned}$$

Similarly, assume  $m \geq 3$  and  $\lfloor \frac{m}{2\Delta} \rfloor < i \leq m$ . We have  $m \leq 2\Delta i$ . Thus, we have

$$\begin{aligned}
& [u^{\Delta i}] (1 + w_2 u^\Delta)^{\Delta^2 m} \\
&= \binom{\Delta^2 m}{i} w_2^i \\
(52) \quad & \text{(by (50))} \leq \left( \frac{e\Delta^2 m}{i} \right)^i w_2^i \\
& \text{(by } m \leq 2\Delta i \text{)} \leq (2e\Delta^3 w_2)^i \\
& \text{(by } w_2 = (4e\Delta^3)^{-1} \text{)} = 2^{-i}.
\end{aligned}$$

Combining (49) with (51) and (52), we have if  $m \geq 3$ ,

$$[x^m]h(x) \leq m^{-1} \left( \sum_{i=0}^{\lfloor m/\Delta \rfloor} 2^{1-m+\Delta i-i} \right) \leq 2^{1-\lfloor m/\Delta \rfloor}.$$

For the case when  $m = 1$  and  $m = 2$ , one can also verify  $[x^m]h(x) \leq 2^{1-\lfloor m/\Delta \rfloor}$  directly from (49). Therefore, by  $f_1 = w_1 h$  and  $w_1 = (8ek\Delta)^{-1}$ , we have

$$[x^m]f_1(x) = w_1 [x^m]h(x) \leq 2^{-\lfloor m/\Delta \rfloor - 1} \cdot \Delta^{-1}.$$

Combining with (48), (47) is immediate.

In the following, we present the injection from the generalized  $\{2, 3\}$ -trees in  $\mathcal{T}_v^t$  to the  $\{a, b\}$ -labelled trees in  $\mathcal{T}_a^t$  as claimed. Then the lemma is proved. Given integers  $r, \ell \geq 0$  and  $z \in \{a, b\}$ , define  $\mathcal{T}_z^{r, \ell} \triangleq \{\text{proper subtree } T \text{ of } \mathcal{T}_z \mid (a(T) = r) \wedge (b(T) = \ell)\}$ . To construct an injection from  $\mathcal{T}_v^t$  to  $\mathcal{T}_a^t$  such that each  $T \in \mathcal{T}_v^t$  is mapped to some  $T' \in \mathcal{T}_a^t$  satisfying  $|T \cap V| = a(T')$  and  $|T \cap \mathcal{E}| = b(T')$ , it is sufficient to construct an injection from  $\mathcal{T}_v^{r, \ell}$  to  $\mathcal{T}_a^{r, \ell}$  for each integers  $r, \ell \geq 0$ . In the following, we present the injection from  $\mathcal{T}_v^{r, \ell}$  to  $\mathcal{T}_a^{r, \ell}$ . Given a generalized  $\{2, 3\}$ -trees  $T \in \mathcal{T}_v^{r, \ell}$ , by Definition 7.1 we have  $G(T)$  has a rooted directed spanning tree with root  $v$ . Choose such a spanning tree  $\phi(T)$  arbitrarily. Obviously,  $\phi(T) \neq \phi(T')$  for different  $T, T' \in \mathcal{T}_v^t$ . Thus,  $\phi$  is an injection from the set  $\mathcal{T}_v^{r, \ell}$  to the set  $\{\phi(T) \mid T \in \mathcal{T}_v^{r, \ell}\}$ . For each node  $u$  in  $\phi(T)$ , we have either  $u \in V$  or  $u \in \mathcal{E}$ . In addition, by  $T \in \mathcal{T}_v^{r, \ell}$ , we have  $|T \cap V| = r$ ,  $|T \cap \mathcal{E}| = \ell$ . Combining with  $\phi(T)$  is a spanning tree of  $G(T)$ , we have the node set of  $\phi(T)$  is  $T$  and then  $\#\{\text{nodes of } \phi(T) \text{ in } V\} = r$ ,  $\#\{\text{nodes of } \phi(T) \text{ in } \mathcal{E}\} = \ell$ .

Given  $H = (V, \mathcal{E})$  and  $u \in V$ , let  $V(u) \subseteq V$  be the set of vertices  $u'$  such that there exists  $e \in \mathcal{E}$  satisfying  $u, u' \in e$ , and  $\mathcal{E}(u) \subseteq \mathcal{E}$  be the set of hyperedges  $e$  such that there exists  $e' \in \mathcal{E}$  such that  $u \in e' \wedge \text{dist}_{\text{Lin}(H)}(e', e) = 1$ . Similarly, given  $e \in \mathcal{E}$ , let  $V(e) \subseteq V$  be the set of vertices  $u$  such that there exists  $e' \in \mathcal{E}$  satisfying  $u \in e' \wedge \text{dist}_{\text{Lin}(H)}(e, e') = 1$  or  $2$ , and  $\mathcal{E}(e) \subseteq \mathcal{E}$  be the set of hyperedges  $e'$  such that  $\text{dist}_{\text{Lin}(H)}(e, e') = 2$  or  $3$ . Moreover, recall that each hyperedge in  $H$  contains at most  $k$  vertices and shares vertices with at most  $\Delta$  hyperedges. We have for each  $u \in V$  and  $e \in \mathcal{E}$ ,  $|V(u)| \leq k\Delta$ ,  $|\mathcal{E}(u)| \leq \Delta^2$ ,  $|V(e)| \leq k\Delta^2$ ,  $|\mathcal{E}(e)| \leq \Delta^3$ . Let  $\mathbb{T}_v$  be a rooted directed tree with infinite nodes such that

- each node is labelled with some  $u \in V \cup \mathcal{E}$  and the root is labelled  $v$ ;
- for each  $u \in V \cup \mathcal{E}$  and  $u' \in V(u) \cup \mathcal{E}(u)$ , each node labelled  $u$  has exactly one child labelled  $u'$ .

Given integers  $r, \ell \geq 0$ , let  $\mathbb{T}_v^{r, \ell}$  be the set of subtrees  $T$  of  $\mathbb{T}_v$  such that the root of  $T$  is the root of  $\mathbb{T}_v$  and  $\#\{\text{nodes in } T \text{ with labels from } V\} = r$ ,  $\#\{\text{nodes in } T \text{ with labels from } \mathcal{E}\} = \ell$ . In addition, by the definition of  $\mathbb{T}_v$ , one can verify that

- the root of  $\mathbb{T}_v$  has a label from  $V$ ;
- each node labelled some  $u \in V$  has  $|V(u)| \leq k\Delta$  children with labels from  $V$  and  $|\mathcal{E}(u)| \leq \Delta^2$  children with labels from  $\mathcal{E}$ ;
- each node labelled some  $u \in \mathcal{E}$  has  $|V(u)| \leq k\Delta^2$  children with labels from  $V$  and  $|\mathcal{E}(u)| \leq \Delta^3$  children with labels from  $\mathcal{E}$ .

Combining with the definitions of  $\mathcal{T}_a$ ,  $\mathcal{T}_a^{r, \ell}$  and  $\mathbb{T}_v^{r, \ell}$ , one can verify that  $|\mathbb{T}_v^{r, \ell}| \leq |\mathcal{T}_a^{r, \ell}|$ . Thus, there exists an injection  $\psi$  from  $\mathbb{T}_v^{r, \ell}$  to  $\mathcal{T}_a^{r, \ell}$ .

In addition, by Definition 7.1, for each  $T \in \mathcal{T}_v^{r, \ell}$  and each arc from  $u$  to  $u'$  in  $G(T)$ , we have  $u \in T \subseteq V \cup \mathcal{E}$  and  $u' \in V(u) \cup \mathcal{E}(u)$ . Combining with that  $\phi(T)$  is a rooted directed spanning tree of  $G(T)$ , we have for each node  $u$  in  $\phi(T)$ , the children of  $u$  are from  $V(u) \cup \mathcal{E}(u)$ . Thus,  $\phi(T)$  is a subtree of  $\mathbb{T}_v$ . Combining with that  $v$  is the root of  $\phi(T)$  and  $\#\{\text{nodes of } \phi(T) \text{ in } V\} = r$ ,  $\#\{\text{nodes of } \phi(T) \text{ in } E\} = \ell$ , we have  $\phi(T) \in \mathbb{T}_v^{r, \ell}$ . Formally,  $\{\phi(T) \mid T \in \mathcal{T}_v^{r, \ell}\} \subseteq \mathbb{T}_v^{r, \ell}$ . Recall that  $\phi$  is an injection from the set  $\mathcal{T}_v^{r, \ell}$  to the set  $\{\phi(T) \mid T \in \mathcal{T}_v^{r, \ell}\}$ . We have  $\phi$  is an injection from  $\mathcal{T}_v^{r, \ell}$  to  $\mathbb{T}_v^{r, \ell}$ . Recall that  $\psi$  is an injection from  $\mathbb{T}_v^{r, \ell}$  to  $\mathcal{T}_a^{r, \ell}$ . We have  $\psi \circ \phi$  is an injection from  $\mathcal{T}_v^{r, \ell}$  to  $\mathcal{T}_a^{r, \ell}$ . Thus the lemma follows.  $\square$

Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment such that only one variable  $v \in V$  has  $\sigma(v) = \star$ . The following lemma bounds the length  $\ell$  of  $\text{Path}(\sigma) = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$  and further shows that if the set of bad variables/constraints given  $\sigma_\ell$  becomes too large, then there exists a large generalized  $\{2, 3\}$ -tree  $T$  in  $H_\Phi$  such that the event  $\mathcal{E}_T^\sigma$  as in Definition 6.13 happens.

**Lemma 7.6.** *Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment with exactly one variable  $v \in V$  having  $\sigma(v) = \star$ , and let  $\text{Path}(\sigma) = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ . Suppose  $|V_\star^{\sigma_\ell}| + |C_{\star\text{-frozen}}^{\sigma_\ell}| \geq L$  for some integer  $L \geq 1$ , then there exists a generalized  $\{2, 3\}$ -tree  $T = U \uplus E$  of  $H_\Phi$  with root  $v$  such that  $\mathcal{E}_T^\sigma$  happens and  $|U| + \Delta \cdot |E| \geq L$ .*

Before proving Lemma 7.6, we show we can already prove Lemma 6.15 using Lemma 7.6.

*Proof of Lemma 6.15.* The case  $i = 0$  is trivial. In the following, we assume  $i > 0$ . Recall  $V = \{v_1, v_2, \dots, v_n\}$ . We claim that for each possible  $\sigma$ , given  $X_0^t = \sigma$ , if  $|V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta$ , then  $\mathcal{E}_T^t$  happens for some  $L \geq i\Delta$  and  $T \in \mathcal{T}_{v_t}^L$ . Formally,

$$(53) \quad \Pr \left[ \left( |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right) \wedge (X_0^t = \sigma) \right] \leq \Pr \left[ \left( \exists L \geq i\Delta, T \in \mathcal{T}_{v_t}^L \text{ s.t. } \mathcal{E}_T^t \text{ happens} \right) \wedge (X_0^t = \sigma) \right].$$

Combining with the union bound, we have

$$(54) \quad \Pr \left[ \left( |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right) \wedge (X_0^t = \sigma) \right] \leq \sum_{L \geq i\Delta} \sum_{T \in \mathcal{T}_{v_t}^L} \Pr \left[ \mathcal{E}_T^t \wedge (X_0^t = \sigma) \right].$$

Therefore, we have

$$(55) \quad \begin{aligned} & \Pr \left[ |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right] \cdot \mathbb{E} \left[ \chi(X_0^t) \mid |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right] \\ &= \Pr \left[ |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right] \cdot \sum_{\sigma} \left( \chi(\sigma) \Pr \left[ X_0^t = \sigma \mid |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right] \right) \\ &= \sum_{\sigma} \left( \chi(\sigma) \Pr \left[ (X_0^t = \sigma) \wedge \left( |V_\star^{X_0^t}| + |C_{\star\text{-frozen}}^{X_0^t}| \geq i\Delta \right) \right] \right) \\ &\leq \sum_{\sigma} \chi(\sigma) \left( \sum_{L \geq i\Delta} \sum_{T \in \mathcal{T}_{v_t}^L} \Pr \left[ \mathcal{E}_T^t \wedge (X_0^t = \sigma) \right] \right) = \sum_{L \geq i\Delta} \sum_{T \in \mathcal{T}_{v_t}^L} \sum_{\sigma} \left( \chi(\sigma) \Pr \left[ \mathcal{E}_T^t \wedge (X_0^t = \sigma) \right] \right), \end{aligned}$$

where the first equality is by the law of total expectation, the second equality is by the chain rule, and the inequality is by the non-negativity of  $\chi(\cdot)$  and (54). Moreover, by the law of total expectation, we

also have

$$(56) \quad \sum_{\sigma} (\chi(\sigma) \Pr [\mathcal{E}_T^t \wedge (X_0^t = \sigma)]) = \sum_{\sigma} (\chi(\sigma) \Pr [X_0^t = \sigma | \mathcal{E}_T^t] \Pr [\mathcal{E}_T^t]) \leq \Pr [\mathcal{E}_T^t] \mathbb{E} [\chi(X_0^t) | \mathcal{E}_T^t].$$

Thus, we have

$$\begin{aligned} & \Pr \left[ \left| V_{\star}^{X_0^t} \right| + \left| C_{\star\text{-frozen}}^{X_0^t} \right| \geq i\Delta \right] \cdot \mathbb{E} \left[ \chi(X_0^t) \mid \left| V_{\star}^{X_0^t} \right| + \left| C_{\star\text{-frozen}}^{X_0^t} \right| \geq i\Delta \right] \\ \text{(by (55) and (56))} &= \sum_{L \geq i\Delta} \sum_{T \in \mathcal{T}_{v_t}^L} (\Pr [\mathcal{E}_T^t] \cdot \mathbb{E} [\chi(X_0^t) | \mathcal{E}_T^t]) \\ &= \sum_{j \geq i} \sum_{r=0}^{\Delta-1} \sum_{T \in \mathcal{T}_{v_t}^{j\Delta+r}} (\Pr [\mathcal{E}_T^t] \cdot \mathbb{E} [\chi(X_0^t) | \mathcal{E}_T^t]) \\ \text{(by Lemma 7.2)} &\leq \sum_{j \geq i} \sum_{r=0}^{\Delta-1} \sum_{T \in \mathcal{T}_{v_t}^{j\Delta+r}} \left( (8ek\Delta)^{-|T \cap V|} \cdot (4e\Delta^3)^{-|T \cap \mathcal{C}|} \right) \\ \text{(by Lemma 7.4)} &\leq \sum_{j \geq i} \sum_{r=0}^{\Delta-1} (2^{-j-1} \cdot \Delta^{-1}) \\ &= 2^{-i}. \end{aligned}$$

In the following, we prove (53). Then the theorem is proved. Given a possible assignment  $\sigma$  of  $X_0^t$ , suppose  $X_0^t = \sigma$  and  $\left| V_{\star}^{X_0^t} \right| + \left| C_{\star\text{-frozen}}^{X_0^t} \right| \geq i\Delta$ . Then, by (25) we have  $\left| V_{\star}^{\sigma} \right| + \left| C_{\star\text{-frozen}}^{\sigma} \right| \geq i\Delta$ . We claim  $\sigma = X_{v_t \leftarrow \star}^{t-1}$ . In addition, we have  $X^{t-1}(v) \neq \star$  for any  $v \in V$ , because by (25),  $X^{t-1}$  is generated from  $X^0 = \star^V$  with Algorithm 5 and no vertex in Algorithm 5 is set as  $\star$ . Combining with  $\sigma = X_{v_t \leftarrow \star}^{t-1}$  and (25), we have there exists only one variable  $v_t \in V$  such that  $\sigma(v_t) = \star$ . Combining with Lemma 7.6 and  $\left| V_{\star}^{\sigma} \right| + \left| C_{\star\text{-frozen}}^{\sigma} \right| \geq i\Delta$ , we have  $\mathcal{E}_T^{\sigma}$  happens for some  $L \geq i\Delta$  and  $T \in \mathcal{T}_{v_t}^L$ . In addition, by  $X_0^t = \sigma$  and the definitions of  $\mathcal{E}_T^t$  and  $\mathcal{E}_T^{\sigma}$  in Definition 6.13, we have  $\mathcal{E}_T^{\sigma}$  is exact  $\mathcal{E}_T^t$ . Therefore,  $\mathcal{E}_T^t$  happens for some  $L \geq i\Delta$  and  $T \in \mathcal{T}_{v_t}^L$  and (53) is immediate.

At last, we show the claim  $\sigma = X_{v_t \leftarrow \star}^{t-1}$  to finish the proofs of (53) and the lemma. By (25) we have either  $\sigma = X_0^t = X^{t-1}$  or  $\sigma = X_0^t = X_{v_t \leftarrow \star}^{t-1}$ . Thus, it is sufficient to show  $\sigma \neq X^{t-1}$ . Suppose  $\sigma = X^{t-1}$  for contradiction. Recall that  $X^{t-1}(v) \neq \star$  for any  $v \in V$ . We have  $\sigma(v) = X^{t-1}(v) \neq \star$  for any  $v \in V$ . Thus, by Definition 6.10 we have  $\text{Path}(\sigma) = \sigma$  and  $\sigma_{\ell} = \sigma$ . Thus,  $\sigma_{\ell}(v) \neq \star$  for any  $v \in V$ . Combining with Definition 6.13, we have  $V_{\star}^{\sigma_{\ell}} = \emptyset$  and  $C_{\star\text{-frozen}}^{\sigma_{\ell}} = \emptyset$ , which is contradictory with  $\left| V_{\star}^{\sigma} \right| + \left| C_{\star\text{-frozen}}^{\sigma} \right| \geq i\Delta$ . Thus, we have  $\sigma \neq X^{t-1}$  and then  $\sigma = X_{v_t \leftarrow \star}^{t-1}$ . This finishes the proofs of (53) and the lemma.  $\square$

To prove Lemma 7.6, we need to introduce the definition of  $G_{VC}$ , an undirected graph with a vertex set over all variables and constraints of the CSP formula.

**Definition 7.7** (Graph of variables and constraints). Let  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  be the CSP formula. Define  $G_{VC} = (V \cup \mathcal{C}, E)$  as the graph where vertices are  $V \cup \mathcal{C}$  and there is an edge between two vertices  $u, v$  if and only if one of the following holds:

- (1)  $u, v \in V$  and there exists some  $c \in \mathcal{C}$  such that  $u, v \in \text{vbl}(c)$ .
- (2)  $u, v \in \mathcal{C}$  and  $\text{dist}_{\text{Lin}(H_{\Phi})}(u, v) = 1$  or  $2$ .
- (3)  $u \in V, v \in \mathcal{C}$  and there exists some  $c \in \mathcal{C}$  such that  $u \in \text{vbl}(c) \wedge \text{dist}_{\text{Lin}(H_{\Phi})}(c, v) = 1$ .

Furthermore, for any  $S \subseteq V \cup \mathcal{C}$ , we let  $G_{VC}(S)$  denote the subgraph of  $G_{VC}$  induced by  $S$ .

Recall in Definition 3.6: for any  $\sigma \in \mathcal{Q}^*$ ,  $H_{\text{fix}}^{\sigma}$  denotes the sub-hypergraph of  $H^{\sigma}$  induced by  $V^{\sigma} \cap V_{\text{fix}}^{\sigma}$ . Recall that for each  $v \in V^{\sigma}$ ,  $H_v^{\sigma} = (V_v^{\sigma}, \mathcal{C}_v^{\sigma})$  denotes the connected component in  $H^{\sigma}$  that contains the vertex/variable  $v$ . Also, for each  $c \in \mathcal{C}$ , we denote the simplified constraint of  $c$  under  $\sigma$  as  $c^{\sigma}$ .

The following lemma states a connectivity property on the graph  $G_{VC}$ .

**Lemma 7.8.** *Assume the condition of Lemma 7.6. Then  $G_{VC} \left( C_{\star\text{-frozen}}^{\sigma_i} \cup V_{\star}^{\sigma_i} \right)$  is connected for each  $0 \leq i \leq \ell$ .*

*Proof.* We prove this lemma by induction on  $i$ . For simplicity, we say a variable or constraint  $c$  is connected to a subset  $S \subseteq V \cup \mathcal{C}$  in  $G_{VC}$  if  $c$  is connected to some  $c' \in S$ . The base case is when  $i = 0$ . By the condition of the lemma,  $v$  is the only variable satisfying  $\sigma(v) = \star$ . Combining with  $\sigma_0 = \sigma$ , we have  $v$  is the only variable satisfying  $\sigma_0(v) = \star$ . Therefore,  $V_{\star}^{\sigma_0} = \{v\}$ . In addition, we have the following claim: each  $c \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_0}$  is connected to  $v$  in  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_0} \cup V_{\star}^{\sigma_0})$ . Combining with the claim, we have  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_0} \cup V_{\star}^{\sigma_0})$  is connected.

Now we prove the claim, which completes the proof of the base case. By  $c \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_0}$ , we have  $c \in \mathcal{C}_{\star\text{-con}}^{\sigma_0} \cap \mathcal{C}_{\text{frozen}}^{\sigma_0}$ . By  $c \in \mathcal{C}_{\star\text{-con}}^{\sigma_0}$  and Definition 3.6, we have  $V_{\star\text{-con}}^{\sigma_0} \cap \text{vbl}(c) \neq \emptyset$ . Combining with  $v$  is the only variable satisfying  $\sigma_0(v) = \star$  and the definition of  $V_{\star\text{-con}}^{\sigma_0}$ , we have there exists a connected path  $c_1^{\sigma_0}, c_2^{\sigma_0}, \dots, c_t^{\sigma_0} = c^{\sigma_0} \in \mathcal{C}^{\sigma_0}$  such that  $\sigma_0(v) = \star, v \in \text{vbl}(c_1^{\sigma_0})$  and  $\text{vbl}(c_j^{\sigma_0}) \subseteq V^{\sigma_0} \cap V_{\text{fix}}^{\sigma_0}$  for each  $j < t$ . If  $c = c_1$ , then  $v \in \text{vbl}(c)$  and the claim is immediate by the definition of  $G_{VC}$ . In the following, we assume  $c \neq c_1$ . Let  $w_j \in (\text{vbl}(c_j^{\sigma_0}) \cap \text{vbl}(c_{j+1}^{\sigma_0}))$  for each  $j < t$ . Then  $w_j \notin \Lambda(\sigma_0)$ . By  $w_j \in \text{vbl}(c_j^{\sigma_0})$  and  $\text{vbl}(c_j^{\sigma_0}) \subseteq V_{\text{fix}}^{\sigma_0}$ , we have  $w_j \in V_{\text{fix}}^{\sigma_0}$ . Combining with  $w_j \notin \Lambda(\sigma_0)$ , we have either  $\sigma_0(w_j) = \star$ , where we set  $\widehat{c}_j = w_j$ ; or  $w_j \in \text{vbl}(\widehat{c}_j)$  for some  $\widehat{c}_j \in \mathcal{C}_{\text{frozen}}^{\sigma_0}$ . Note that  $\widehat{c}_j$  can be either a variable or a constraint. In the former case, we have  $\widehat{c}_j \in V_{\star}^{\sigma_0}$ . In the latter case, By  $w_j$  is connected to  $v$  in  $H_{\text{fix}}^{\sigma_0}$  through the path  $c_1^{\sigma_0}, c_2^{\sigma_0}, \dots, c_j^{\sigma_0}$ , we have  $w_j \in V_{\star\text{-con}}^{\sigma_0}$ . Thus, we have  $\widehat{c}_j \in \mathcal{C}_{\star\text{-con}}^{\sigma_0}$  by Definition 3.6. Combining with  $\widehat{c}_j \in \mathcal{C}_{\text{frozen}}^{\sigma_0}$ , we have  $\widehat{c}_j \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_0}$ . In summary, we always have  $\widehat{c}_j \in V_{\star}^{\sigma_0} \cup \mathcal{C}_{\star\text{-frozen}}^{\sigma_0}$ . Moreover, for each  $j < t - 1$ , if  $\widehat{c}_j \in \mathcal{C}$ , we have  $w_j \in \text{vbl}(c_{j+1}^{\sigma_0}) \cap \text{vbl}(\widehat{c}_j^{\sigma_0})$ , otherwise we have  $\widehat{c}_j = w_j$ . Thus by Definition 7.7, it can be verified that  $\widehat{c}_j$  and  $\widehat{c}_{j+1}$  are adjacent in  $G_{VC}$ . In addition, if  $\widehat{c}_1 \in \mathcal{C}$ , we have  $w_1 \in \text{vbl}(c_1) \cap \text{vbl}(\widehat{c}_1)$ , otherwise we have  $\text{vbl}(\widehat{c}_1) = w_1 \in \text{vbl}(c_1)$ , hence  $c_1$  and  $\widehat{c}_1$  are adjacent in  $G_{VC}$ . Similarly, we have  $\widehat{c}_{t-1}$  and  $c_t$  are adjacent in  $G_{VC}$ . Thus, we have  $v, c_1, \widehat{c}_1, \widehat{c}_2, \dots, \widehat{c}_{t-1}, c_t = c$  is a connected path in  $G_{VC}$ . Combining with  $v \in V_{\star}^{\sigma_0}$  and  $\widehat{c}_j \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_0}$  for each  $j < t$ , the claim is immediate.

For the induction step, we prove this lemma for each  $i > 0$ . We claim that each  $v \in V_{\star}^{\sigma_i}$  is connected to  $V_{\star}^{\sigma_{i-1}}$  in  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_i} \cup V_{\star}^{\sigma_i})$ . In addition, we can prove each  $c \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_i}$  is connected to  $V_{\star}^{\sigma_{i-1}}$  in  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_i} \cup V_{\star}^{\sigma_i})$  by a similar argument to the base case. Moreover, by the induction hypothesis we have  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}} \cup V_{\star}^{\sigma_{i-1}})$  is connected. Combining with  $\mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}} \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_i}$  and  $V_{\star}^{\sigma_{i-1}} \subseteq V_{\star}^{\sigma_i}$  by Lemma 6.14, we have  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_i} \cup V_{\star}^{\sigma_i})$  is connected.

Now we prove the claim that each  $v \in V_{\star}^{\sigma_i}$  is connected to  $V_{\star}^{\sigma_{i-1}}$  in  $G_{VC}$ , which completes the proof of the lemma. If  $v \in V_{\star}^{\sigma_{i-1}}$ , the claim is immediate by  $V_{\star}^{\sigma_{i-1}} \subseteq V_{\star}^{\sigma_i}$ . In the following, we assume  $v \in V_{\star}^{\sigma_i} \setminus V_{\star}^{\sigma_{i-1}}$ , where by Definition 6.10 we have  $v = \text{NextVar}(\sigma_{i-1})$ . By the definition of  $\text{NextVar}(\cdot)$ , we have  $v \in V_{\star\text{-inf}}^{\sigma_{i-1}}$  and then  $v \in \text{vbl}(\widehat{c})$  for some constraint  $\widehat{c} \in \mathcal{C}_{\star\text{-con}}^{\sigma_{i-1}}$ . In addition, by  $\widehat{c} \in \mathcal{C}_{\star\text{-con}}^{\sigma_{i-1}}$  one can verify that there exists a variable  $w \neq v$  and a connected path  $c_1^{\sigma_{i-1}}, c_2^{\sigma_{i-1}}, \dots, c_t^{\sigma_{i-1}} = \widehat{c}^{\sigma_{i-1}} \in \mathcal{C}^{\sigma_{i-1}}$  such that  $\sigma_{i-1}(w) = \star, w \in \text{vbl}(c_1^{\sigma_{i-1}})$  and  $\text{vbl}(c_j^{\sigma_{i-1}}) \subseteq V^{\sigma_{i-1}} \cap V_{\text{fix}}^{\sigma_{i-1}}$  for each  $j < t$ . Then there are two possibilities for  $\widehat{c}$ .

- If  $\widehat{c} = c_1$ , we have  $v, w \in \text{vbl}(c_1)$ . Therefore,  $v$  is connected to  $w$  in  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}} \cup V_{\star}^{\sigma_{i-1}} \cup \{v\})$ . Also by  $\sigma_{i-1}(w) = \star$  we have  $w \in V_{\star}^{\sigma_{i-1}}$ . In addition, by Lemma 6.14 we have

$$\mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}} \cup V_{\star}^{\sigma_{i-1}} \cup \{v\} \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}} \cup V_{\star}^{\sigma_i} \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_i} \cup V_{\star}^{\sigma_i}.$$

Thus the claim is immediate.

- Otherwise,  $\widehat{c} \neq c_1$ . Similarly to the base case, one can find a connected path  $c_1, \widehat{c}_1, \widehat{c}_2, \dots, \widehat{c}_{t-1}, c_t = \widehat{c}$  in  $G_{VC}$ , where  $w \in \text{vbl}(c_1)$ ,  $\widehat{c}_j \in V_{\star}^{\sigma_{i-1}} \cup \mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}}$  for each  $j < t$ , and there exists  $w_{t-1} \in \text{vbl}(c_t) \cap \text{vbl}(\widehat{c}_{t-1})$ . Recall that  $v \in \text{vbl}(\widehat{c})$  and  $w \in \text{vbl}(c_1)$ . Thus,  $w, c_1, \widehat{c}_1, \widehat{c}_2, \dots, \widehat{c}_{t-1}, v$  is also a connected path in  $G_{VC}$ . Combining with  $w \in V_{\star}^{\sigma_{i-1}}$ ,  $\widehat{c}_j \in V_{\star}^{\sigma_{i-1}} \cup \mathcal{C}_{\star\text{-frozen}}^{\sigma_{i-1}}$  for each  $j < t$ , we have  $v$  is connected to  $V_{\star}^{\sigma_{i-1}}$  in  $G_{VC}(\mathcal{C}_{\star\text{-frozen}}^{\sigma_i} \cup V_{\star}^{\sigma_i})$ . Thus the claim holds.  $\square$

The following crucial technical lemma states that when the set of “bad” variables and constraints is large, there exists a large generalized  $\{2, 3\}$ -tree capturing the occurrence of such event.

**Lemma 7.9.** *Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment. If  $G_{\text{VC}} \left( \mathcal{C}_{\star\text{-frozen}}^\sigma \cup V_\star^\sigma \right)$  is connected, then there always exists a generalized  $\{2, 3\}$ -tree  $T = U \uplus E$  in  $H_\Phi$  with root  $v$  such that*

$$U = V_\star^{\sigma_t}, \quad E \subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma, \quad \text{and} \quad \Delta \cdot |E| \geq |\mathcal{C}_{\star\text{-frozen}}^\sigma|$$

*Proof.* We construct a generalized  $\{2, 3\}$ -tree  $T$  and a rooted directed tree  $T^*$  as follows. Let  $B$  denote a subset of variables and constraints and  $R$  denote a subset of constraints. For simplicity, let  $G_{\text{VC}}$  denote  $G_{\text{VC}} \left( \mathcal{C}_{\star\text{-frozen}}^\sigma \cup V_\star^\sigma \right)$ .

- Initially, let  $T = \{v\}$ ,  $B = \mathcal{C}_{\star\text{-frozen}}^\sigma \cup V_\star^\sigma \setminus \{v\}$ ,  $R = \emptyset$ , and let  $T^*$  be a tree with only one vertex  $v$ ;
- while  $B \neq \emptyset$ 
  - Choose  $u \in T$ ,  $w \in B$  according to the following:
    - (i) If there exists  $u \in T$ ,  $w \in B$  such that  $\text{dist}_{G_{\text{VC}}}(u, w) = 1$ , choose any such  $(u, w)$ ;
    - (ii) Otherwise, if there exists  $u \in T \cap \mathcal{C}$ ,  $w \in B$  such that there exists  $c \in \mathcal{C}$  satisfying  $\text{vbl}(u) \cap \text{vbl}(c) \neq \emptyset$  and  $\text{dist}_{G_{\text{VC}}}(c, w) = 1$ , choose any such  $(u, w)$ ;
    - (iii) Otherwise, choose any  $u \in T$ ,  $w \in B$ .
  - add  $w$  to  $T$ , and add a node  $w$  and an arc from  $u$  to  $w$  to  $T^*$ .
  - update  $B, R$  as follows:
    - (a) If  $w \in V$ , update  $B \leftarrow B \setminus \{w\}$ ,  $R \leftarrow R$
    - (b) If  $w \in \mathcal{C}$ , update  $B \leftarrow B \setminus \Gamma(w)$ ,  $R \leftarrow R \cup \Gamma(w)$ , where  $\Gamma(w) = \{c \in \mathcal{C} \mid \text{vbl}(w) \cap \text{vbl}(c) \neq \emptyset\}$ .

Let  $U = T \cap V$  and  $E = T \cap \mathcal{C}$ . We claim that when the above construction process stops,  $T = U \uplus E$  is a generalized  $\{2, 3\}$ -tree in  $H_\Phi$  with root  $v$  satisfying  $U = V_\star^\sigma$ ,  $E \subseteq \mathcal{C}_{\text{frozen}}^\sigma$  and  $\Delta \cdot |E| \geq |\mathcal{C}_{\text{frozen}}^\sigma|$ .

We first show that  $T$  is a generalized  $\{2, 3\}$ -tree in  $H_\Phi$  with root  $v$ . From the construction process, we know each  $w \in \mathcal{C}_{\star\text{-frozen}}^\sigma \cup V_\star^\sigma$  is either added into  $T$  or removed from  $B$  in Item (b) when some  $c \in \mathcal{C}$  is added into  $T$  and  $w \in \Gamma(c)$ . We will simply refer to the latter case as “removed in Item (b)” for the rest of the proof. If  $w$  is removed in Item (b), then  $w \in \mathcal{C}$  and there exists  $c \in \mathcal{C}$  such that  $\text{vbl}(c) \cap \text{vbl}(w) \neq \emptyset$  and  $c$  is added into  $T$ . Therefore for all distinct  $u, v \in T \cap \mathcal{C}$ , we have  $\text{dist}_{\text{Lin}(H_\Phi)}(u, v) \geq 2$ . Thus, by Definition 7.1, to show  $T$  is a generalized  $\{2, 3\}$ -tree in  $H_\Phi$ , it is sufficient to show the claim that  $T^*$  is a rooted spanning tree of  $G(T)$ , where  $G(T)$  is defined in Definition 7.1. In the next, we prove this claim. By the construction process, we have  $T^*$  is a rooted connected tree with the node set  $T$  immediately. Thus, it is sufficient to show that each arc of  $T^*$  is in  $G(T)$ . For each  $u \in T \setminus \{v\}$ , let  $w$  be the only father of  $u$  in  $T^*$ . In other words, there is an arc from  $w$  to  $u$ . Then when the pair  $u, w$  is chosen in the construction process, we have the following cases:

- $\text{dist}_{G_{\text{VC}}}(u, w) = 1$ . This corresponds to the case of Item (i). In this case, if either  $u \in V$  or  $w \in V$ , by comparing Definition 7.7 with the definition of  $G(T)$  in Definition 7.1, one can verify that the arc from  $u$  to  $w$  must be an arc of  $G(T)$ . Otherwise, we have  $u, w \in \mathcal{C}$ . By Item (b) of the construction process, we have  $\text{dist}_{\text{Lin}(H_\Phi)}(u, w) \geq 2$ . By Definition 7.7 and  $\text{dist}_{G_{\text{VC}}}(u, w) = 1$ , we have  $\text{dist}_{\text{Lin}(H_\Phi)}(w, u) = 2$ . Thus, one can also verify that the arc from  $w$  to  $u$  is an arc of  $G(T)$  by Definition 7.1.
- Otherwise,  $\text{dist}_{G_{\text{VC}}}(u, w) > 1$ . Then the condition in Item (i) is not satisfied, otherwise, some  $u' \in T$ ,  $w' \in B$  where  $\text{dist}_{G_{\text{VC}}}(u', w') = 1$  rather than  $u, w$  will be chosen in the process. Thus, we have  $\text{dist}_{G_{\text{VC}}}(T, B) \triangleq \min_{a \in T, b \in B} \text{dist}_{G_{\text{VC}}}(a, b) > 1$  when the pair  $u, w$  is chosen in the construction process. Moreover, it is straightforward from the construction process that the three sets  $T, R$  and  $B$  form a partition of the vertex set of  $G_{\text{VC}}$ . In addition, we have  $G_{\text{VC}}$  is connected by assumption. Therefore  $\text{dist}_{G_{\text{VC}}}(T \cup R, B) = 1$ . Combining with  $\text{dist}_{G_{\text{VC}}}(T, B) > 1$ , we have  $\text{dist}_{G_{\text{VC}}}(R, B) = 1$  and there exist  $c' \in R$ ,  $w' \in B$  where  $\text{dist}_{G_{\text{VC}}}(c', w') = 1$ . By  $c' \in R$  and Item (b) of the construction process, there must be some  $u' \in \mathcal{C}$  such that  $c' \in \Gamma(u')$  and  $c'$  is removed from  $B$  in Item (b) when  $u' \in \mathcal{C}$  is added into  $T$ . Thus, we have  $\text{vbl}(c') \cap \text{vbl}(u') \neq \emptyset$  and  $u'$  has been added to  $T$  when the pair  $u, w$  is chosen in the construction process. Thus,  $u', w'$  satisfy the condition in Item (ii). Therefore, we have  $u, w$  also satisfy the condition in Item (ii), otherwise,  $u', w'$  rather than  $u, w$  will be chosen in the process. Thus, we have  $u \in T \cap \mathcal{C}$  and there exists  $c \in \mathcal{C}$  satisfying  $\text{vbl}(u) \cap \text{vbl}(c) \neq \emptyset$  and  $\text{dist}_{G_{\text{VC}}}(c, w) = 1$ . If  $w \in \mathcal{C}$ , by  $\text{dist}_{G_{\text{VC}}}(c, w) = 1$  and

Definition 7.7, we have  $\text{dist}_{\text{Lin}(H_\Phi)}(c, w) = 1$  or  $2$ . Combining with  $\text{vbl}(u) \cap \text{vbl}(c) \neq \emptyset$ , we have  $\text{dist}_{\text{Lin}(H_\Phi)}(u, w) = 1, 2$  or  $3$ . Combining with  $\text{dist}_{G_{\text{VC}}}(u, w) > 1$ , we have  $\text{dist}_{\text{Lin}(H_\Phi)}(u, w) = 3$ . Otherwise,  $\text{dist}_{\text{Lin}(H_\Phi)}(u, w) = 1$  or  $2$ . By Definition 7.7, we have  $\text{dist}_{G_{\text{VC}}}(u, w) = 1$ , which is a contradiction. Combining  $u, w \in \mathcal{C}$ ,  $\text{dist}_{\text{Lin}(H_\Phi)}(u, w) = 3$  with Definition 7.1, we have the arc from  $u$  to  $w$  is also an arc of  $G(T)$ . If  $w \in V$ , by  $\text{dist}_{G_{\text{VC}}}(c, w) = 1$  and Definition 7.7, there exists some  $c' \in \mathcal{C}$  such that  $w \in \text{vbl}(c') \wedge \text{dist}_{\text{Lin}(H_\Phi)}(c, c') = 1$ . Combining with  $\text{vbl}(u) \cap \text{vbl}(c) \neq \emptyset$ , we have  $w \in \text{vbl}(c') \wedge \text{dist}_{\text{Lin}(H_\Phi)}(u, c') = 1$  or  $2$ . Combining with Definition 7.1, we also have the arc from  $u$  to  $w$  is an arc of  $G(T)$ .

This shows that  $T^*$  is a directed spanning tree of  $G(T)$ . In addition, it is easy to verify that  $v$  is the root of  $T^*$ . Therefore,  $T$  is a generalized  $\{2, 3\}$ -tree in  $H_\Phi$  with root  $v$ .

At last, we show  $U = V_{\star}^\sigma$  and  $E \subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma$ . By the construction process, one can verify that  $U = V_{\star}^\sigma$ . Moreover, by Item (b) of the process, at most  $|\Gamma(w)| \leq \Delta - 1$  constraints are moved from  $B$  when a constraint  $w \in \mathcal{C} \cap B$  is added to  $T$ . Combining with  $\mathcal{C}_{\star\text{-frozen}}^\sigma \subseteq B$  in the initialization, we have  $\Delta \cdot |E| \geq |\mathcal{C}_{\star\text{-frozen}}^\sigma|$ . This completes the proof.  $\square$

Now we are ready to prove Lemma 7.6.

*Proof of Lemma 7.6.* By  $\sigma \in \mathcal{Q}^*$  is a partial assignment with exactly one variable  $v \in V$  having  $\sigma(v) = \star$  and Lemma 7.8, we have  $G_{\text{VC}}\left(\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell} \cup V_{\star}^{\sigma_\ell}\right)$  is connected. Combining with Lemma 7.9, there exists a generalized  $\{2, 3\}$ -tree  $T = U \uplus E$  in  $H_\Phi$  such that  $U = V_{\star}^{\sigma_\ell}$ ,  $E \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}$  and  $\Delta \cdot |E| \geq |\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}|$ . It is straightforward to verify that  $\Delta \cdot |E| + |U| \geq L$  by the assumption that  $|V_{\star}^{\sigma_\ell}| + |\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}| \geq L$ .  $\square$

We then prove Lemma 6.16. Recall the definition of generalized  $\{2, 3\}$ -tree in Definition 7.1 and the definition of  $H_\Phi = (V, \mathcal{C})$  in Section 3.2. We have the following lemma which is similar to Lemma 7.6.

**Lemma 7.10.** *For every  $v \in V$ , there exists a generalized  $\{2, 3\}$ -tree  $T = \{v\} \uplus E$  in  $H_\Phi$  with root  $v$  where  $E \subseteq \mathcal{C}_{\text{frozen}}^{X^n}$  and  $\Delta^2 |E| \geq |\mathcal{C}_v^{X^n}|$ .*

Before proving Lemma 7.10, we show we can already prove Lemma 6.16 using Lemma 7.10.

*Proof of Lemma 6.16.* The case  $i = 0$  is trivial. In the following, we assume  $i > 0$ . Recall the definition of  $H_\Phi = (V, \mathcal{C})$  in Section 3.2. Given integers  $t, r, \ell \geq 0$  and  $v \in V$ , recall  $\mathcal{F}_v^t$  and  $\mathcal{F}_v^{r, \ell}$  in Definition 7.1. For any  $i \geq 1$  and  $v \in V$ , we have

$$\begin{aligned}
& \Pr\left[|\mathcal{C}_v^X| \geq 2i\Delta^2\right] \\
\text{(by Lemma 7.10)} & \leq \sum_{j \geq 2i} \sum_{T \in \mathcal{F}_v^{1, j}} \Pr\left[T \cap \mathcal{C} \subseteq \mathcal{C}_{\text{frozen}}^X\right] \\
\text{(by Lemma 7.3)} & \leq \sum_{j \geq 2i} \sum_{T \in \mathcal{F}_v^{1, j}} (4e\Delta^3)^{-|T \cap \mathcal{C}|} \\
(57) & = 8ek\Delta \sum_{j \geq 2i} \sum_{T \in \mathcal{F}_v^{1, j}} (8ek\Delta)^{-1} (4e\Delta^3)^{-|T \cap \mathcal{C}|} \\
& \text{(by } \mathcal{F}_v^{1, j} \subseteq \mathcal{F}_v^{j\Delta+1}\text{)} \leq 8ek\Delta \sum_{j \geq 2i} \sum_{T \in \mathcal{F}_v^{j\Delta+1}} (8ek\Delta)^{-|T \cap V|} (4e\Delta^3)^{-|T \cap \mathcal{C}|} \\
& \text{(by Lemma 7.4)} \leq 8ek\Delta \cdot \sum_{j \geq 2i} \left(2^{-j-1} \cdot \Delta^{-1}\right) \\
& \leq 8ek \cdot 4^{-i}.
\end{aligned}$$

$\square$

We then finish Section 7.1 by proving Lemma 7.10. Let  $X = X^n$  where  $X^0, X^1, \dots, X^n$  is the partial assignment sequence of Algorithm 1 in Definition 5.6. Then we have the following lemma.

**Lemma 7.11.**  $V \setminus \Lambda(X) \subseteq \text{vbl}\left(\mathcal{C}_{\text{frozen}}^X\right)$ .

*Proof.* Given  $v_i \in V \setminus \Lambda(X)$  where  $i \in [n]$ , by Lines 2-4 of Algorithm 1 and Theorem 5.5, we have  $v_i \in V_{\text{fix}}^{X^{i-1}}$ , otherwise,  $v_i$  will be assigned a value from  $Q_{v_i}$  in Line 4. Moreover, again by Lines 2-4 of Algorithm 1, we also have  $v_i \notin \Lambda^+(X^{i-1})$ . Combining with  $v_i \in V_{\text{fix}}^{X^{i-1}}$ , we have  $v_i \in \mathcal{C}_{\text{frozen}}^{X^{i-1}}$ . In addition, similar to Lemma B.2, one can also prove that  $\mathcal{C}_{\text{frozen}}^{X^{j-1}} \subseteq \mathcal{C}_{\text{frozen}}^{X^j}$  for each  $j \in [n]$ . Then we have  $\mathcal{C}_{\text{frozen}}^{X^{i-1}} \subseteq \mathcal{C}_{\text{frozen}}^X$  by induction. Combining with  $v_i \in \mathcal{C}_{\text{frozen}}^{X^{i-1}}$ , we have  $v_i \in \text{vbl}(\mathcal{C}_{\text{frozen}}^X)$ . Then the lemma follows.  $\square$

Now we can prove Lemma 7.10.

*Proof of Lemma 7.10.* Let  $\{\Phi_i^X = (V_i^X, \mathcal{C}_i^X) \mid 1 \leq i \leq K\}$  be the decomposition of  $\Phi^X$ . If  $v \notin V_i$  for each  $i \in [k]$ , we have  $\mathcal{C}_v^X = \emptyset$  and the lemma is trivial. In the following, we assume w.l.o.g. that  $v \in V_i^X$  for some  $i \in K$ . Then we have  $\Phi_v^X = (V_i^X, \mathcal{C}_i^X)$ . Let

$$S \triangleq \{c \in \mathcal{C}_{\text{frozen}}^X \mid c^X \in \mathcal{C}_i^X\}.$$

At first, we prove that there exists some  $c_v \in S$  such that  $v \in \text{vbl}(c_v)$ . By  $v \in V_i^X$ , we have  $v \notin \Lambda(X)$ . Combining with Lemma 7.11, we have there exists some  $c_v \in \mathcal{C}_{\text{frozen}}^X$  such that  $v \in \text{vbl}(c_v)$ . In addition, by  $v \in \text{vbl}(c_v)$  and  $c_v \in \mathcal{C}_{\text{frozen}}^X$ , we also have  $c_v^X \in \mathcal{C}_i^X$ . Combining with  $c_v \in \mathcal{C}_{\text{frozen}}^X$ , we have  $c_v \in S$ .

Now we prove  $|\mathcal{C}_i^X| \leq \Delta |S|$ . For each  $c^X \in \mathcal{C}_i^X$ , we have there exists a connected path  $c_1^X, c_2^X, \dots, c_t^X = c^X \in \mathcal{C}_i^X$  such that  $v \in \text{vbl}(c_1^X)$ . Let  $v' \in \text{vbl}(c^X)$ . We have  $v' \notin \Lambda(X)$ . Combining with Lemma 7.11, we have  $v' \in \text{vbl}(\widehat{c})$  for some  $\widehat{c} \in \mathcal{C}_{\text{frozen}}^X$ . Then we have  $\widehat{c}^X \in \mathcal{C}_i^X$  because there exists a connected path  $c_1^X, c_2^X, \dots, c_t^X, \widehat{c}^X \in \mathcal{C}_i^X$  where  $v \in \text{vbl}(c_1^X)$ . Combining  $\widehat{c} \in \mathcal{C}_{\text{frozen}}^X$  with  $\widehat{c}^X \in \mathcal{C}_i^X$ , we have  $\widehat{c} \in S$ . In summary, for each  $c^X \in \mathcal{C}_i^X$ , there exists some  $\widehat{c} \in S$  such that  $\text{vbl}(c^X) \cap \text{vbl}(\widehat{c}^X) \neq \emptyset$ . Thus, we have  $|\mathcal{C}_i^X| \leq \Delta |S|$ .

In the next, we prove that  $G_{\text{VC}}(S)$  is connected. It is enough to prove that any two different constraints  $c, \widehat{c} \in S$  are connected in  $G_{\text{VC}}(S)$ . Given  $c, \widehat{c} \in S$ , we have  $c^X, \widehat{c}^X$  are in  $\mathcal{C}_i^X$ . Therefore, we have there exists a connected path  $c^X = c_1^X, c_2^X, \dots, c_t^X = \widehat{c}^X \in \mathcal{C}_i^X$ . If  $t \leq 3$ , obviously  $c$  and  $\widehat{c}$  are connected in  $G^2(S)$ . In the following, we assume that  $t > 3$ . Let  $w_j \in (\text{vbl}(c_j^X) \cap \text{vbl}(c_{j+1}^X))$  for each  $j < t$ . Then we have  $w_j \notin \Lambda(X)$ . Combining with Lemma 7.11, we have  $w_j \in \text{vbl}(\widehat{c}_j^X)$  for some  $\widehat{c}_j \in \mathcal{C}_{\text{frozen}}^X$ . Moreover, we also have  $\widehat{c}^X \in \mathcal{C}_i^X$ , because  $\widehat{c}_j^X$  is connected to  $c^X$  through  $c_2^X, \dots, c_j^X \in \mathcal{C}_i^X$ . Thus, we have  $\widehat{c}_j \in S$ . In addition, for each  $\widehat{c}_j, \widehat{c}_{j+1}$  where  $j < t-1$ , we have  $\widehat{c}_j$  and  $\widehat{c}_{j+1}$  are connected in  $G^2(\mathcal{C})$ , because  $w_j \in \text{vbl}(\widehat{c}_j) \cap \text{vbl}(c_{j+1})$  and  $w_{j+1} \in \text{vbl}(\widehat{c}_{j+1}) \cap \text{vbl}(c_{j+1})$ . Thus, the constraints  $c = c_1, \widehat{c}_1, \widehat{c}_2, \dots, \widehat{c}_{t-1}, c_t = \widehat{c}$  forms a connected path in  $G_{\text{VC}}$ . Combining with  $\widehat{c}_j \in S$  for each  $j \leq t-1$  and  $c, \widehat{c} \in S$ , we have the constraints  $c, \widehat{c}$  are connected in  $G_{\text{VC}}(S)$ .

In summary, we have  $c_v \in S \subseteq \mathcal{C}_{\text{frozen}}^X$ ,  $\Delta |S| \geq |\mathcal{C}_i^X|$  and  $G_{\text{VC}}(S)$  is connected. Combining with  $v \in \text{vbl}(c_v)$  we have  $G_{\text{VC}}(S \cup \{v\})$  is also connected. By going through the process in the proof of Lemma 7.9, we have there exists a subset of constraints and variables  $T \subseteq S \cup \{v\}$  such that  $T = \{v\} \uplus E$  is a generalized  $\{2, 3\}$ -tree in  $H_\Phi$  with root  $v$  and

$$|E| \geq |S| / \Delta \geq |\mathcal{C}_i^X| / \Delta^2 = |\mathcal{C}_v^X| / \Delta^2.$$

In addition, if  $|\mathcal{C}_v^X| \geq L\Delta$ , then it is straightforward to verify that  $\Delta \cdot |E| + 1 \geq L$  and the lemma follows.  $\square$

**7.2. Basic tail bounds for bad events.** In this subsection, we prove Lemma 7.2 and Lemma 7.3. For any constraint  $c \in \mathcal{C}$  and partial assignment  $\sigma \in \mathcal{Q}^*$ , let  $Z(\sigma, c) \triangleq |\text{vbl}(c) \setminus \Lambda(\sigma)|$  denote the number of unassigned variables in  $\text{vbl}(c)$  under  $\sigma$ . For any subset of variables and constraints  $T = U \uplus E$  and partial assignment  $\sigma \in \mathcal{Q}^*$ , define

$$(58) \quad g(\sigma, T) \triangleq \prod_{v \in U \setminus V_\star^\sigma} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[-c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right).$$

The following lemma is the core of Lemma 7.2, which shows that  $g(\star^V, T)$  is an upper bound of the left hand of (46).

**Lemma 7.12.** Given  $t, (X^0, X^1, \dots, X^{t-1}, X_0^t, X_1^t, \dots, X_\ell^t)$  and  $T = U \uplus E$  as in Lemma 7.2, for any partial assignment  $\sigma \in \mathcal{Q}^*$  and any integer  $0 \leq i \leq t-1$  where  $\Pr[X^i = \sigma] > 0$ , we have

$$(59) \quad \Pr[\mathcal{E}_T^t \mid X^i = \sigma] \cdot \mathbb{E}[\chi(X_0^t) \mid \mathcal{E}_T^t \wedge X^i = \sigma] \leq g(\sigma, T).$$

Specifically,

$$(60) \quad \Pr[\mathcal{E}_T^t] \cdot \mathbb{E}[\chi(X_0^t) \mid \mathcal{E}_T^t] \leq g(\star^V, T).$$

The following lemma is a special case of (59), given  $i = t-1$  and  $r_t = 1$  where  $r_t$  is defined as in (25).

**Lemma 7.13.** Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment satisfying  $\mathbb{P}[\neg c \mid \sigma] \leq \alpha q$  for all  $c \in C$ . For any subset of variables and constraints  $T = U \uplus E$  such that the constraints in  $E$  are disjoint,

$$(61) \quad \Pr[\mathcal{E}_T^\sigma] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma] \leq g(\sigma, T).$$

The following lemma provides a useful recursion of  $g(\cdot, \cdot)$ , which is used in the proof of Lemma 7.13.

**Lemma 7.14.** Under the condition of Lemma 7.13, if  $\text{NextVar}(\sigma) = u \neq \perp$ , then

$$\sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot g(\sigma_{u \leftarrow x}, T)) \leq g(\sigma, T)$$

*Proof.* To prove this lemma, it is sufficient to show that

$$(62) \quad \begin{aligned} & \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{v \in U \setminus V_\star^{\sigma_{u \leftarrow x}}} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \\ & \leq \prod_{v \in U \setminus V_\star^\sigma} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right) \end{aligned}$$

Combining with (58), we have

$$\begin{aligned} & \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot g(\sigma_{u \leftarrow x}, T)) \\ \text{(by (58))} & = \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{v \in U \setminus V_\star^{\sigma_{u \leftarrow x}}} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \\ \text{(by (62))} & \leq \prod_{v \in U \setminus V_\star^\sigma} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right) \\ \text{(by (58))} & = g(\sigma, T). \end{aligned}$$

The lemma is proved. In the following, we prove (62).

In addition, by the constraints in  $E$  are disjoint, we have  $\text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset$  for any different  $c, c' \in E$ . Thus, there exists at most one unique constraint  $c_0 \in E$  such that  $u \in \text{vbl}(c_0)$ . Let  $S = E \setminus \{c_0\}$  if  $u \in \text{vbl}(E)$  and  $S = E$  otherwise. Thus for each  $c \in S$ , we have  $u \notin \text{vbl}(c)$ . Then  $\mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] = \mathbb{P}[\neg c \mid \sigma]$  and  $Z(\sigma_{u \leftarrow x}, c) = Z(\sigma, c)$  for each  $x \in Q_u$ . Therefore,

$$(63) \quad \begin{aligned} & \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \\ & = \prod_{c \in S} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{c \in E \setminus S} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \\ & = \prod_{c \in S} \left( \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right) \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{c \in E \setminus S} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right). \end{aligned}$$



In addition, by Corollary 4.3 and the assumption that  $\mathbb{P}[\neg c \mid \sigma] \leq \alpha q$  for all  $c \in \mathcal{C}$ , we have for each  $x \in Q_u$ ,  $\mu_u^\sigma(x) \leq q_u^{-1}(1 + \eta)$ . Therefore,

$$\sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot \mathbb{P}[\neg c_0 \mid \sigma_{u \leftarrow x}]) \leq (1 + \eta) \cdot q_u^{-1} \sum_{x \in Q_u} \mathbb{P}[\neg c_0 \mid \sigma_{u \leftarrow x}] = (1 + \eta) \cdot \mathbb{P}[\neg c_0 \mid \sigma].$$

Thus, we have

$$\begin{aligned} \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \mathbb{P}[\neg c_0 \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c_0)} \right) &= \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \mathbb{P}[\neg c_0 \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma, c_0) - 1} \right) \\ &\leq \mathbb{P}[\neg c_0 \mid \sigma] (1 + \eta)^{Z(\sigma, c_0)}. \end{aligned}$$

Therefore, if  $E \setminus S = \{c_0\}$ , we have

$$(64) \quad \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{c \in E \setminus S} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \leq \prod_{c \in E \setminus S} \left( \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right).$$

If  $E \setminus S = \emptyset$ , both sides of (64) are equal to 1 and we also have (64), where we assume that a product over an empty set is 1. Because  $E \setminus S$  is either  $\{c_0\}$  or  $\emptyset$ , we always have (64). Combining with (63), we have

$$(65) \quad \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \leq \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right).$$

Moreover, by  $u = \text{NextVar}(\sigma)$ , we have  $\sigma(u) = \star \neq \star$ . Thus,  $u \notin V_\star^\sigma$ . Meanwhile, by  $\sigma_{u \leftarrow x}(u) = x \neq \star$ , we also have  $u \notin V_\star^{u \leftarrow x}$  for each  $x \in Q_u$ . Thus,  $U \setminus V_\star^\sigma = U \setminus V_\star^{u \leftarrow x}$ . Combining with (65), we have

$$\begin{aligned} &\sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{v \in U \setminus V_\star^{u \leftarrow x}} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \\ &= \left( \prod_{v \in U \setminus V_\star^\sigma} (1 - q_v \cdot \theta_v) \right) \cdot \sum_{x \in Q_u} \left( \mu_u^\sigma(x) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] (1 + \eta)^{Z(\sigma_{u \leftarrow x}, c)} \right) \right) \\ &\leq \prod_{v \in U \setminus V_\star^\sigma} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right) \end{aligned}$$

Then (62) and the lemma are proved.  $\square$

Now we can prove Lemma 7.13.

*Proof of Lemma 7.13.* Let  $\text{Path}(\sigma) = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ . We show the lemma by a structural induction on  $\text{Path}(\sigma)$ . The base case is when  $\ell = 0$ . By Definition 6.10, we have  $\sigma_\ell = \sigma_0 = \sigma$ . In this case,  $\mathcal{E}_T^\sigma$  is the deterministic event  $U = V_\star^\sigma \wedge E \subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma$ . If  $U \neq V_\star^\sigma$  or  $E \not\subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma$ , we have  $\Pr[\mathcal{E}_T^\sigma] = 0$ . In addition, by (58) one can verify that  $g(\sigma, T) \geq 0$ . Then the lemma is immediate. Otherwise, we have  $U = V_\star^\sigma \wedge E \subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma$  and the event  $\mathcal{E}_T^\sigma$  happens. By (19) and  $\sigma = \sigma_\ell$ , we have  $\chi(\sigma) = \chi(\sigma_\ell, \sigma) = \chi(\sigma, \sigma) = 1$ . Thus,

$$(66) \quad \Pr[\mathcal{E}_T^\sigma] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma] = \Pr[\mathcal{E}_T^\sigma] = 1.$$

Meanwhile, by  $E \subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma$  and Definition 6.13, we have  $E \subseteq \mathcal{C}_{\star\text{-frozen}}^\sigma \subseteq \mathcal{C}_{\text{frozen}}^\sigma$ . Thus, each  $c \in E$  is  $\sigma$ -frozen. By Remark 3.2, we have  $\mathbb{P}[\neg c \mid \sigma] \geq 0.99\alpha$ . Thus,

$$(67) \quad (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \geq (0.99\alpha)^{-1} \cdot 0.99\alpha \cdot (1 + \eta)^{Z(\sigma, c)} \geq 1.$$

In addition, by (58) and  $U = V_\star^\sigma$ , we have

$$g(\sigma, T) = \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right).$$

Combining with (66) and (67), we have

$$g(\sigma, T) \geq 1 \geq \Pr[\mathcal{E}_T^\sigma] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma].$$

The base case is proved.

For the induction step, we assume that  $\ell(\sigma) \geq 1$ , which by Item 1 of Definition 6.10, says that  $\text{NextVar}(\sigma) = u \neq \perp$  for some  $u \in V$ . According to Item 2 of Definition 6.10, we have

$$\forall x \in \mathcal{Q}_u^*, \quad \Pr[\sigma_1 = \sigma_{u \leftarrow x}] = \psi_u^\sigma(x).$$

Thus, by the law of total probability, we have

$$\begin{aligned} & \Pr[\mathcal{E}_T^\sigma] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma] \\ (68) \quad &= \sum_{x \in \mathcal{Q}_u^*} (\Pr[\sigma_1 = \sigma_{u \leftarrow x}] \cdot \Pr[\mathcal{E}_T^\sigma \mid \sigma_1 = \sigma_{u \leftarrow x}] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})]) \\ &= \sum_{x \in \mathcal{Q}_u^*} (\psi_u^\sigma(x) \cdot \Pr[\mathcal{E}_T^\sigma \mid \sigma_1 = \sigma_{u \leftarrow x}] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})]). \end{aligned}$$

Moreover, by (19) we have

$$\begin{aligned} (69) \quad & \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})] = \mathbb{E}[\chi(\sigma_\ell, \sigma_0) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})] \\ &= (2 - q_u \cdot \theta_u) \mathbb{E}[\chi(\sigma_\ell, \sigma_1) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})]. \end{aligned}$$

In addition, for each  $x \in \mathcal{Q}_u^*$ , given  $\sigma_1 = \tau \triangleq \sigma_{u \leftarrow x}$ , we have the subsequence  $(\sigma_1, \sigma_2, \dots, \sigma_\ell)$  is identically distributed as  $\text{Path}(\tau)$  by the Markov property of the Path process. Thus, we have  $\sigma_\ell$  is identically distributed as  $\tau_{\ell(\tau)}$ . Combining with the definition of  $\mathcal{E}_T^\sigma$  in Definition 6.13, we have

$$\mathbb{E}[\chi(\sigma_\ell, \sigma_1) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \tau)] = \mathbb{E}[\chi(\tau_{\ell(\tau)}, \tau) \mid \mathcal{E}_T^\tau] = \mathbb{E}[\chi(\tau) \mid \mathcal{E}_T^\tau].$$

Combining with (69), we have

$$\begin{aligned} \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})] &= (2 - q_u \cdot \theta_u) \mathbb{E}[\chi(\sigma_\ell, \sigma_1) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})] \\ &= (2 - q_u \cdot \theta_u) \mathbb{E}[\chi(\sigma_\ell, \sigma_1) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \tau)] \\ &= (2 - q_u \cdot \theta_u) \mathbb{E}[\chi(\tau) \mid \mathcal{E}_T^\tau] \\ &= (2 - q_u \cdot \theta_u) \mathbb{E}[\chi(\sigma_{u \leftarrow x}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow x}}]. \end{aligned}$$

Combining with (68), we have

$$\begin{aligned} & \Pr[\mathcal{E}_T^\sigma] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma] \\ &= \sum_{x \in \mathcal{Q}_u^*} (\psi_u^\sigma(x) \cdot \Pr[\mathcal{E}_T^\sigma \mid \sigma_1 = \sigma_{u \leftarrow x}] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma \wedge (\sigma_1 = \sigma_{u \leftarrow x})]) \\ &= (2 - q_u \cdot \theta_u) \sum_{x \in \mathcal{Q}_u^*} (\psi_u^\sigma(x) \cdot \Pr[\mathcal{E}_T^\sigma \mid \sigma_1 = \sigma_{u \leftarrow x}] \cdot \mathbb{E}[\chi(\sigma_{u \leftarrow x}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow x}}]). \end{aligned}$$

In addition, for each  $x \in \mathcal{Q}_u^*$ , recall that  $\sigma_\ell$  is identically distributed as  $\sigma_{u \leftarrow x}$  given  $\sigma_1 = \sigma_{u \leftarrow x}$ . Combining with Definition 6.13, we have  $\Pr[\mathcal{E}_T^\sigma \mid \sigma_1 = \sigma_{u \leftarrow x}] = \Pr[\mathcal{E}_T^{\sigma_{u \leftarrow x}}]$ . Therefore,

$$(70) \quad \Pr[\mathcal{E}_T^\sigma] \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma] = (2 - q_u \cdot \theta_u) \sum_{x \in \mathcal{Q}_u^*} (\psi_u^\sigma(x) \Pr[\mathcal{E}_T^{\sigma_{u \leftarrow x}}] \mathbb{E}[\chi(\sigma_{u \leftarrow x}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow x}}]).$$

We then show the induction step for two cases respectively, namely the case when  $u \in U$  and the case when  $u \notin U$ . At first we assume  $u \in U$ . Given  $x \in \mathcal{Q}_u$  and  $\tau = \sigma_{u \leftarrow x}$ , by  $\tau(u) = x$ , we also have  $\tau_{\ell(\tau)}(u) = x \neq \star$ . Thus  $u \notin V_\star^{\tau_{\ell(\tau)}}$ . Combining with  $u \in U$ , we have  $U \neq V_\star^{\tau_{\ell(\tau)}}$ . Combining with Definition 6.13, we have  $\mathcal{E}_T^\tau$  does not happen. In summary, for each  $x \in \mathcal{Q}_u$ ,  $\mathcal{E}_T^{\sigma_{u \leftarrow x}}$  does not happen and  $\Pr[\mathcal{E}_T^{\sigma_{u \leftarrow x}}] = 0$ . Combining with (70) and (18), we have

$$\begin{aligned} (71) \quad & \Pr[\mathcal{E}_T^\sigma] \cdot \mathbb{E}[\chi(\sigma) \mid \mathcal{E}_T^\sigma] = (2 - q_u \cdot \theta_u) \cdot \psi_u^\sigma(\star) \cdot \Pr[\mathcal{E}_T^{\sigma_{u \leftarrow \star}}] \cdot \mathbb{E}[\chi(\sigma_{u \leftarrow \star}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow \star}}] \\ &= (1 - q_u \cdot \theta_u) \cdot \Pr[\mathcal{E}_T^{\sigma_{u \leftarrow \star}}] \cdot \mathbb{E}[\chi(\sigma_{u \leftarrow \star}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow \star}}]. \end{aligned}$$

In addition, by  $\sigma \in \mathcal{Q}^*$  is a partial assignment satisfying  $\mathbb{P}[\neg c \mid \sigma] \leq \alpha q$  for all  $c \in \mathcal{C}$ , one can also verify  $\mathbb{P}[\neg c \mid \sigma_{u \leftarrow x}] \leq \alpha q$  for all  $c \in \mathcal{C}$  and  $x \in \mathcal{Q}_u^*$  by a similar argument as Lemma 5.9. Thus by the

induction hypothesis, for each  $x \in Q_u^\star$  we have

$$(72) \quad \Pr [\mathcal{E}_T^{\sigma_{u \leftarrow x}}] \cdot \mathbb{E} [\chi(\sigma_{u \leftarrow x}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow x}}] \leq g(\sigma_{u \leftarrow x}, T).$$

Combining with (71) and (58), we have

$$\begin{aligned} & \Pr [\mathcal{E}_T^\sigma] \cdot \mathbb{E} [\chi(\sigma) \mid \mathcal{E}_T^\sigma] \leq (1 - q_u \cdot \theta_u) \cdot g(\sigma_{u \leftarrow \star}, T) \\ & = (1 - q_u \cdot \theta_u) \prod_{v \in U \setminus V_{\star}^{\sigma_{u \leftarrow \star}}} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[-c \mid \sigma_{u \leftarrow \star}] (1 + \eta)^{Z(\sigma_{u \leftarrow \star}, c)} \right) \end{aligned}$$

In addition, by  $u = \text{NextVar}(\sigma)$  and Definition 3.6, we have  $\sigma(u) = \star \neq \star$ . Thus,  $u \notin V_{\star}^\sigma$ . Meanwhile, by  $\sigma_{u \leftarrow \star}(u) = \star$ , we have  $u \in V_{\star}^{u \leftarrow \star}$ . Thus,  $V_{\star}^{\sigma_{u \leftarrow \star}} = V_{\star}^\sigma \uplus \{u\}$ . Combining with  $u \in U$ , we have  $U \setminus V_{\star}^\sigma = (U \setminus V_{\star}^{\sigma_{u \leftarrow \star}}) \uplus \{u\}$ . Therefore,

$$(1 - q_u \cdot \theta_u) \prod_{v \in U \setminus V_{\star}^{\sigma_{u \leftarrow \star}}} (1 - q_v \cdot \theta_v) = \prod_{v \in U \setminus V_{\star}^\sigma} (1 - q_v \cdot \theta_v).$$

Thus, we have

$$\Pr [\mathcal{E}_T^\sigma] \cdot \mathbb{E} [\chi(\sigma) \mid \mathcal{E}_T^\sigma] \leq \prod_{v \in U \setminus V_{\star}^\sigma} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[-c \mid \sigma_{u \leftarrow \star}] (1 + \eta)^{Z(\sigma_{u \leftarrow \star}, c)} \right).$$

By  $\mathbb{P}[-c \mid \sigma_{u \leftarrow \star}] = \mathbb{P}[-c \mid \sigma]$  and  $Z(\sigma_{u \leftarrow \star}, c) = Z(\sigma, c)$  for each  $\sigma \in \mathcal{Q}^*$  and  $c \in C$ , we have

$$\Pr [\mathcal{E}_T^\sigma] \cdot \mathbb{E} [\chi(\sigma) \mid \mathcal{E}_T^\sigma] \leq \prod_{v \in U \setminus V_{\star}^\sigma} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[-c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right) = g(\sigma, T).$$

Then (61) is immediate. This finishes the induction step for the case when  $u \in U$ .

In the following, we assume  $u \notin U$ . Given  $\tau = \sigma_{u \leftarrow \star}$ , by  $\tau(u) = \star$ , we also have  $\tau_{\ell(\tau)}(u) = \star$ . Thus  $u \in V_{\star}^{\tau_{\ell(\tau)}}$ . Combining with  $u \notin U$ , we have  $U \neq V_{\star}^{\tau_{\ell(\tau)}}$ . Combining with Definition 6.13, we have  $\mathcal{E}_T^\tau = \mathcal{E}_T^{\sigma_{u \leftarrow \star}}$  does not happen and  $\Pr [\mathcal{E}_T^{\sigma_{u \leftarrow \star}}] = 0$ . Combining with (70) and (18), we have

$$\begin{aligned} \Pr [\mathcal{E}_T^\sigma] \cdot \mathbb{E} [\chi(\sigma) \mid \mathcal{E}_T^\sigma] & = (2 - q_u \cdot \theta_u) \sum_{x \in Q_u} (\psi_u^\sigma(x) \Pr [\mathcal{E}_T^{\sigma_{u \leftarrow x}}] \mathbb{E} [\chi(\sigma_{u \leftarrow x}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow x}}]) \\ & = \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot \Pr [\mathcal{E}_T^{\sigma_{u \leftarrow x}}] \cdot \mathbb{E} [\chi(\sigma_{u \leftarrow x}) \mid \mathcal{E}_T^{\sigma_{u \leftarrow x}}]). \end{aligned}$$

Combining with (72) and (58), we have

$$\Pr [\mathcal{E}_T^\sigma] \cdot \mathbb{E} [\chi(\sigma) \mid \mathcal{E}_T^\sigma] \leq \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot g(\sigma_{u \leftarrow x}, T)) \leq g(\sigma, T),$$

where the last inequality is by Lemma 7.14. This finishes the induction step for the case when  $u \notin U$ . The lemma is proved.  $\square$

Now we can prove Lemma 7.12 with Lemma 7.13.

*Proof of Lemma 7.12.* To prove the lemma, it is sufficient to prove (59), because (60) is a special case of (59) when  $i = 0$ . In the following, we show (59) by induction on  $i$ . The base case is when  $i = t - 1$ . Conditioning on  $X^i = X^{t-1} = \sigma$ , by (25) we have either  $X_0^t = \sigma$  or  $X_0^t = \sigma_{v_t \leftarrow \star}$ . By  $\Pr [X^i = \sigma] > 0$  we have  $\sigma(v) \neq \star$  for any  $v \in V$ , because by (25),  $X^i$  is generated from  $X^0 = \star^V$  with Algorithm 5 and no variable in Algorithm 5 is set as  $\star$ . If  $X_0^t = \sigma$ , by (25) we have  $X_\ell^t = \text{Path}(X_0^t) = \text{Path}(\sigma)$ . In addition, by Definition 6.10 we have  $\text{Path}(\sigma) = \sigma$ . Thus, we have  $X_\ell^t = \text{Path}(\sigma) = \sigma$ ,  $V_{\star}^{X_\ell^t} = V_{\star}^\sigma$ , and  $C_{\star\text{-frozen}}^{X_\ell^t} = C_{\star\text{-frozen}}^\sigma$ . In addition, by Definition 6.13 and that  $\sigma(v) \neq \star$  for any  $v \in V$ , we have  $V_{\star}^\sigma = \emptyset$  and  $C_{\star\text{-frozen}}^\sigma = \emptyset$ . Thus,  $V_{\star}^{X_\ell^t} = \emptyset$  and  $C_{\star\text{-frozen}}^{X_\ell^t} = \emptyset$ . Combining with the definition of  $\mathcal{E}_T^t$  in Definition 6.13, we have  $\mathcal{E}_T^t$  does not happen. Thus, we have

$$(73) \quad \Pr [\mathcal{E}_T^t \mid X_0^t = \sigma] = \Pr [X_0^t = \sigma \mid \mathcal{E}_T^t] = 0.$$

Therefore, we have

$$\Pr [\mathcal{E}_T^t \mid X^{t-1} = \sigma]$$

$$\begin{aligned}
(\text{by the law of total probability}) &= \Pr [X_0^t = \sigma \mid X^{t-1} = \sigma] \Pr [\mathcal{E}_T^t \mid (X^{t-1} = \sigma) \wedge (X_0^t = \sigma)] \\
&\quad + \Pr [X_0^t = \sigma_{v_t \leftarrow \star} \mid X^{t-1} = \sigma] \Pr [\mathcal{E}_T^t \mid (X^{t-1} = \sigma) \wedge (X_0^t = \sigma_{v_t \leftarrow \star})] \\
(\text{by (73)}) &= \Pr [X_0^t = \sigma_{v_t \leftarrow \star} \mid X^{t-1} = \sigma] \Pr [\mathcal{E}_T^t \mid (X^{t-1} = \sigma) \wedge (X_0^t = \sigma_{v_t \leftarrow \star})] \\
(\text{by (25)}) &\leq \Pr [r_t = 1] \Pr [\mathcal{E}_T^t \mid (X^{t-1} = \sigma) \wedge (X_0^t = \sigma_{v_t \leftarrow \star})] \\
(\text{by Definition 6.13}) &= \Pr [r_t = 1] \Pr [\mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}]
\end{aligned}$$

Similarly, by the law of total expectation we have

$$\begin{aligned}
&\mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge (X^{t-1} = \sigma)] \\
&= \Pr [X_0^t = \sigma \mid \mathcal{E}_T^t \wedge (X^{t-1} = \sigma)] \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge (X_0^t = \sigma) \wedge (X^{t-1} = \sigma)] \\
&\quad + \Pr [X_0^t = \sigma_{v_t \leftarrow \star} \mid \mathcal{E}_T^t \wedge (X^{t-1} = \sigma)] \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge (X_0^t = \sigma_{v_t \leftarrow \star}) \wedge (X^{t-1} = \sigma)] \\
(\text{by (73)}) &= \Pr [X_0^t = \sigma_{v_t \leftarrow \star} \mid \mathcal{E}_T^t \wedge (X^{t-1} = \sigma)] \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge (X_0^t = \sigma_{v_t \leftarrow \star}) \wedge (X^{t-1} = \sigma)] \\
(\text{by Definition 6.13}) &= \Pr [X_0^t = \sigma_{v_t \leftarrow \star} \mid \mathcal{E}_T^t \wedge (X^{t-1} = \sigma)] \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}] \\
&\leq \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}]
\end{aligned}$$

Combining the above two inequalities, we have

$$\begin{aligned}
&\Pr [\mathcal{E}_T^t \mid X^i = \sigma] \cdot \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge X^i = \sigma] \\
&\leq \Pr [r_t = 1] \Pr [\mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}] \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}] \\
(\text{by (25)}) &= (1 - q_{v_t} \cdot \theta_{v_t}) \Pr [\mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}] \mathbb{E} [\chi(\sigma_{v_t \leftarrow \star}) \mid \mathcal{E}_T^{\sigma_{v_t \leftarrow \star}}] \\
(\text{by Lemma 7.13}) &\leq (1 - q_{v_t} \cdot \theta_{v_t}) g(\sigma_{v_t \leftarrow \star}, T) \\
(\text{by (58)}) &\leq g(\sigma, T).
\end{aligned}$$

The base case is proved.

For the induction step, we assume  $i < t - 1$  and prove (59) based on the hypothesis on  $i + 1$ . Given  $X^i = \sigma$ , if  $v_{i+1}$  is  $\sigma$ -fixed, by (25) and Algorithm 5, we have  $X^{i+1} = X^i = \sigma$ . Then (59) holds by the induction hypothesis. Otherwise,  $v_{i+1}$  is not  $\sigma$ -fixed. Let  $u = v_{i+1}$ . According to Algorithm 5 we have

$$(74) \quad \forall x \in Q_u, \quad \Pr [X^{i+1} = \sigma_{u \leftarrow x} \mid X^i = \sigma] = \mu_u^\sigma(x).$$

In addition, recall that Algorithm 5 generates the prefix  $(X^0, X^1, \dots, X^t)$  of the random partial assignments  $X^0, X^1, \dots, X^n$  maintained in Algorithm 1 defined in Definition 5.6. Combining with Lemma 5.8 and we have  $\mathbb{P}[\neg c \mid X^i] \leq \alpha q$  for all  $c \in \mathcal{C}$ . Combining with  $\Pr [X^i = \sigma] > 0$ , we have  $\mathbb{P}[\neg c \mid \sigma] \leq \alpha q$  for all  $c \in \mathcal{C}$ . Combining with Corollary 4.4, we have  $\mu_u^\sigma(x) > 0$  for each  $x \in Q_u$ . Thus, we have

$$\begin{aligned}
&\Pr [X^{i+1} = \sigma_{u \leftarrow x}] \\
(\text{by the chain rule}) &= \Pr [X^i = \sigma] \Pr [X^{i+1} = \sigma_{u \leftarrow x} \mid X^i = \sigma] \\
(\text{by (74)}) &= \Pr [X^i = \sigma] \mu_u^\sigma(x) \\
(\text{by } \Pr [X^i = \sigma] > 0, \mu_u^\sigma(x) > 0) &> 0.
\end{aligned}
\tag{75}$$

Thus by the induction hypothesis, for each  $x \in Q_u$  we have

$$(76) \quad \Pr [\mathcal{E}_T^t \mid X^{i+1} = \sigma_{u \leftarrow x}] \cdot \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge X^{i+1} = \sigma_{u \leftarrow x}] \leq g(\sigma_{u \leftarrow x}, T).$$

Combining with (74) we have

$$\begin{aligned}
& \Pr [\mathcal{E}_T^t \mid X^i = \sigma] \cdot \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t \wedge X^i = \sigma] \\
\text{(by (74) and (76))} & \leq \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot g(\sigma_{u \leftarrow x}, T)) \\
\text{(by Lemma 7.14)} & \leq g(\sigma, T),
\end{aligned}$$

which finishes the induction step. Then (59) and the lemma are proved.  $\square$

Now we can prove Lemma 7.2.

*Proof of Lemma 7.2.* For each  $c \in E$ , we have  $Z(\star^V, c) = |\text{vbl}(c) \setminus \Lambda(\star^V)| \leq k$ . Combining with (58) we have

$$\begin{aligned}
(77) \quad g(\star^V, T) &= \prod_{v \in V} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \star^V] (1 + \eta)^{Z(\star^V, c)} \right) \\
&\leq \prod_{v \in V} (1 - q_v \cdot \theta_v) \prod_{c \in E} \left( (0.99\alpha)^{-1} p (1 + \eta)^k \right).
\end{aligned}$$

In addition, by (6) and (7), we have  $\eta \leq (2k\Delta)^{-1}$  and

$$(78) \quad \forall v \in V, \quad 1 - q_v \cdot \theta_v \leq 1 - q\theta \leq (8ek\Delta)^{-1}.$$

By  $8ep\Delta^3 \leq 0.99\alpha$  and  $\eta \leq (2k\Delta)^{-1}$ , we have

$$(79) \quad (0.99\alpha)^{-1} p (1 + \eta)^k \leq (4e\Delta^3)^{-1}.$$

Combining (77) with (78) and (79), we have

$$g(\star^V, T) \leq (8ek\Delta)^{-|U|} \cdot (4e\Delta^3)^{-|E|}.$$

Combining with Lemma 7.12, we have

$$\Pr [\mathcal{E}_T^t] \cdot \mathbb{E} [\chi(X_0^t) \mid \mathcal{E}_T^t] \leq g(\star^V, T) \leq (8ek\Delta)^{-|U|} \cdot (4e\Delta^3)^{-|E|}.$$

The lemma is proved.  $\square$

We finish Section 7.2 by proving Lemma 7.3. Recall that (79) holds by (6), (7) and  $8ep\Delta^3 \leq 0.99\alpha$ . Thus Lemma 7.3 is immediate by the following lemma, which is an analogy of Lemma 7.13.

**Lemma 7.15.** *Recall the definition of  $g(\cdot, \cdot)$  in (58). Let  $(X^0, X^1, \dots, X^n) = \text{Simulate}(n)$ . For any  $0 \leq i \leq n$ , any partial assignment  $\sigma \in Q^*$  where  $\Pr [X^i = \sigma] > 0$  and any set of disjoint constraints  $T \subseteq \mathcal{C}$ , we have*

$$(80) \quad \Pr [T \subseteq \mathcal{C}_{\text{frozen}}^{X^n} \mid X^i = \sigma] \leq g(\sigma, T).$$

*Specifically,*

$$(81) \quad \Pr [T \subseteq \mathcal{C}_{\text{frozen}}^{X^n}] \leq g(\star^V, T) = \prod_{c \in T} \left( (0.99\alpha)^{-1} (1 + \eta)^k \right).$$

*Proof.* To prove the lemma, it is sufficient to prove (80), because (81) is a special case of (80) when  $i = 0$ . We show (80) by induction on  $i$ . The base case is when  $i = n$ . For each  $\sigma$ , conditioning on  $X^i = \sigma$ , we have  $X^n = X^i = \sigma$ . Thus, if  $T \not\subseteq \mathcal{C}_{\text{frozen}}^\sigma$ , we have  $\Pr [T \subseteq \mathcal{C}_{\text{frozen}}^{X^n} \mid X^i = \sigma] = \Pr [T \subseteq \mathcal{C}_{\text{frozen}}^\sigma \mid X^i = \sigma] = 0$  and (80) is immediate by the non-negativity of  $g(\cdot, \cdot)$ . Otherwise, we have  $T \subseteq \mathcal{C}_{\text{frozen}}^\sigma$ . According to Remark 3.2, we have  $c$  is  $\sigma$ -frozen only if  $\mathbb{P}[\neg c \mid \sigma] \geq 0.99\alpha$ . Therefore, we have

$$\begin{aligned}
g(\sigma, T) &= \prod_{c \in T} \left( (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \sigma] (1 + \eta)^{Z(\sigma, c)} \right) \geq \prod_{c \in T} \left( (0.99\alpha)^{-1} \mathbb{P}[\neg c \mid \sigma] \right) \\
&\geq \prod_{c \in T} \left( (0.99\alpha)^{-1} \cdot 0.99\alpha \right) \geq 1 = \Pr [T \subseteq \mathcal{C}_{\text{frozen}}^{X^n} \mid X^i = \sigma].
\end{aligned}$$

Thus, we also have (80) and the base case is proved.

For the induction step, we assume  $i < n$  and prove (80) based on the hypothesis on  $i + 1$ . Given  $X^i = \sigma$ , if  $v_{i+1}$  is  $\sigma$ -fixed, by (25) and Algorithm 5, we have  $X^{i+1} = X^i = \sigma$ . Then (80) holds by the induction hypothesis. Otherwise  $v_{i+1}$  is not  $\sigma$ -fixed. Let  $u = v_{i+1}$ . According to Algorithm 5 we have

$$(82) \quad \forall x \in Q_u, \quad \Pr [X^{i+1} = \sigma_{u \leftarrow x} \mid X^i = \sigma] = \mu_u^\sigma(x).$$

Following the proof of (75) in Lemma 7.12, it is easy to verify that  $\Pr [X^{i+1} = \sigma_{u \leftarrow x}] > 0$  for each  $x \in Q_u$ . Thus by the induction hypothesis, for each  $x \in Q_u$  we have

$$(83) \quad \Pr [T \subseteq \mathcal{C}_{\text{frozen}}^{X^n} \mid X^{i+1} = \sigma_{u \leftarrow x}] \leq g(\sigma_{u \leftarrow x}, T).$$

Thus, we have

$$\begin{aligned} & \Pr [T \subseteq \mathcal{C}_{\text{frozen}}^{X^n} \mid X^i = \sigma] \\ \text{(by (82) and (83))} & \leq \sum_{x \in Q_u} (\mu_u^\sigma(x) \cdot g(\sigma_{u \leftarrow x}, T)) \\ \text{(by Lemma 7.14)} & \leq g(\sigma, T), \end{aligned}$$

which finishes the induction step. Then (80) and the lemma are proved.  $\square$

## 8. CONCLUSION AND OPEN PROBLEMS

We give an algorithm for sampling uniform solutions to general constraint satisfaction problems (CSPs) in a local lemma regime. The algorithm runs in an expected near-linear time in the number of variables and polynomial in other local parameters, including: domain size  $q$ , width  $k$  and degree  $\Delta$ .

This gives, for the first time, a near-linear time sampling algorithm for general CSPs with constant  $q, k, \Delta$ , in a local lemma regime; and this also gives, for the first time, a polynomial-time sampling algorithm for general CSPs in a local lemma regime without assuming any degree or width bound.

A crucial step of our sampling algorithm, is a marginal sampler that can draw values of a variable according to the correct marginal distribution. Within a local lemma regime, this marginal sampler is a local algorithm whose cost is independent of the size of the CSP, and is polynomial in the local parameters  $q, k$  and  $\Delta$ . This marginal sampler proves a thought-provoking point: *within a local lemma regime, a locally defined sampling or inference problem can be solved at a local cost.*

There are several open problems:

- An open question is to improve the current LLL condition  $p\Delta^5 \lesssim 1$  for sampling general CSPs closer the lower bound  $p\Delta^2 \gtrsim 1$ . We also believe that removing the extra  $q, k$  factors in the current LLL condition may help us better understand the nature of sampling LLL.
- Another fundamental question is to generalize the current bound for CSPs to a general sampling Lovász local lemma with non-uniform distributions and/or asymmetric criteria.
- The recursive marginal sampler of Anand and Jerrum [AJ22] is a refreshingly novel idea for sampling. Here we see that its can solve an otherwise difficult to solve problem. It would be exciting to see what more this new idea can bring to the study of sampling LLL.
- Despite the current technical barrier, Markov chain based algorithms have several advantages, such as their efficient parallelization [LY22]. Therefore, it is still very worthwhile to have Markov chain based algorithms for sampling general CSPs. For this to work, we may have to develop a way for dynamic projection of solution space, which may be of independent interest.

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We present a generic solution to the following problem. Let  $(\Phi, \sigma, v)$  be the input to Algorithm 4, where  $\Phi = (V, \mathcal{Q}, \mathcal{C})$  is a CSP formula,  $\sigma \in \mathcal{Q}^*$  is a feasible partial assignment, and  $v \in V$  is a variable. Assume Condition 3.8 for the  $(\Phi, \sigma, v)$ . The marginal distribution  $\mu_v^\sigma$  over domain  $Q_v$  is well-defined, and by Corollary 4.4, the following is satisfied for the parameters  $\theta_v > 0$  and  $\zeta > 0$  fixed as in (7):

$$(84) \quad \min_{x \in Q_v} \mu_v^\sigma(x) \geq \theta_v + \zeta.$$

Therefore, the distribution  $\mathcal{D}$  in (8) is well-defined. We reiterate its definition here:

$$(85) \quad \forall x \in Q_v, \quad \mathcal{D}(x) \triangleq \frac{\mu_v^\sigma(x) - \theta_v}{1 - q_v \cdot \theta_v}, \quad \text{where } q_v \triangleq |Q_v|.$$

Our goal is to sample from this distribution by accessing an oracle for drawing independent samples from  $\mu_v^\sigma$ . Such an oracle for  $\mu_v^\sigma$  is realized by `RejectionSampling` $(\Phi, \sigma, \{v\})$  defined in Algorithm 2.

Such a problem of simulating a new coin by making black-box accesses to an old coin, while the distribution of the new coin is a function of the old, is known as the Bernoulli factory problem [von51].

**A.1. Construction and correctness of the Bernoulli factory.** For  $\xi \in [0, 1]$ , we denote by  $\mathcal{O}_\xi$  a coin with probability of heads  $\xi$ . Formally,  $\mathcal{O}_\xi$  is an oracle that, upon each call, independently returns 1 with probability  $\xi$  and 0 with probability  $1 - \xi$ .

We write  $\nu = \mu_v^\sigma$  for short. We construct the following two types of basic oracles:

- for each  $x \in Q_v$ , an  $\mathcal{O}_{\nu(x)}$  is constructed as  $\mathcal{O}_{\nu(x)} = \mathbb{1}[\text{RejectionSampling}(\Phi, \sigma, \{v\}) = x]$ ;
- an  $\mathcal{O}_{\theta_v}$  is constructed as  $\mathcal{O}_{\theta_v} = \mathbb{1}[r < \theta_v]$  for  $r \in [0, 1)$  chosen uniformly at random.

For each  $x \in Q_v$ , we apply the *Bernoulli factory for subtraction* in [NP05], denoted by `SubtractBF`, such that it constructs a new coin  $\mathcal{O}_{\nu(x) - \theta_v} = \text{SubtractBF}(\mathcal{O}_{\nu(x)}, \mathcal{O}_{\theta_v}, \zeta)$  with probability of heads  $\nu(x) - \theta_v$ .

We then apply the *Bernoulli race* in [DHKN17], denoted by `BernoulliRace`, such that the subroutine `BernoulliRace` $(\{\mathcal{O}_{\nu(x) - \theta_v}\}_{x \in Q_v})$  returns a random value  $I \in Q_v$  satisfying that  $I = x$  with probability proportional to  $\nu(x) - \theta_v = \mu_v^\sigma(x) - \theta_v$ , i.e.  $I$  is distributed as  $\mathcal{D}$  defined in (85). This achieves our goal.

Although these constructions are not new, for rigorously, we restate the precise constructions.

The Bernoulli race [DHKN17] subroutine `BernoulliRace` $(\{\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^q\})$  is given accesses to a list of coins  $\mathcal{O}^1 = \mathcal{O}_{\xi_1}, \mathcal{O}^2 = \mathcal{O}_{\xi_2}, \dots, \mathcal{O}^q = \mathcal{O}_{\xi_q}$  with unknown  $\xi_1, \xi_2, \dots, \xi_q \in [0, 1]$ . Its goal is to return a random  $I \in [q]$  such that  $I = i$  with probability  $\xi_i / \sum_{j=1}^q \xi_j$ . This can be achieved by independently repeating the following until a value is returned:

- choose  $I \in [q]$  uniformly at random;
- if a draw of  $\mathcal{O}^I$  returns 1 then return  $I$ .

The correctness of this procedure was given in [DHKN17].

**Proposition A.1** ([DHKN17, Theorem 3.3]). *Given access to a list of coins  $L = \{\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^q\}$ , where for each  $i \in [q]$ , the probability of heads for  $\mathcal{O}^i$  is  $\xi_i$ , the `BernoulliRace` $(L)$  defined above terminates with probability 1 and returns a random  $I \in [q]$  such that  $\Pr[I = i] = \xi_i / \sum_{j=1}^q \xi_j$  for every  $i \in [q]$ .*

To define the Bernoulli factory for subtraction, we further need to construct a linear Bernoulli factory, which transforms  $\mathcal{O}_\xi$  to  $\mathcal{O}_{C\xi}$  for a  $C > 1$  with the promise that  $C\xi \leq 1$ . We adopt the construction of linear Bernoulli factory in [Hub16] described in Algorithm 6. Its correctness is guaranteed as follows.

**Proposition A.2** ([Hub16, Theorem 1]). *Given access to a coin  $\mathcal{O}_\xi$ , given as input  $C > 1$  and  $\zeta > 0$ , with promise that  $C\xi \leq 1 - \zeta$ , `LinearBF` $(\mathcal{O}, C, \zeta)$  terminates with probability 1 and returns a draw of  $\mathcal{O}_{C\xi}$ .*

A Bernoulli factory for subtraction, `SubtractBF` $(\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}, \zeta)$ , is given in [NP05], which transforms two coins  $\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}$  with the promise that  $\xi_1 - \xi_2 \geq \zeta > 0$ , to a new coin  $\mathcal{O}_{\xi_1 - \xi_2}$ . We implement this procedure using the linear Bernoulli factory defined above:

- `SubtractBF` $(\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}, \zeta) = 1 - \text{LinearBF}(\mathcal{O}_{(1 - \xi_1 + \xi_2)/2}, 2, \zeta)$ ,

where the coin  $\mathcal{O}_{(1 - \xi_1 + \xi_2)/2}$  is realized with  $\mathcal{O}_{1/2}, \mathcal{O}_{\xi_1}$  and  $\mathcal{O}_{\xi_2}$  as follows: if  $\mathcal{O}_{1/2} = 1$ , return  $1 - \mathcal{O}_{\xi_1}$ ; otherwise, return  $\mathcal{O}_{\xi_2}$ . The correctness of this procedure is guaranteed as follows.

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**Algorithm 6:** LinearBF( $\mathcal{O}, C, \zeta$ ) [Hub16]

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**Input:** a coin  $\mathcal{O} = \mathcal{O}_\xi$  with unknown  $\xi$ ,  $C > 1$  and a slack  $\zeta > 0$ , with promise that  $C\xi \leq 1 - \zeta$ ;  
**Output:** a random value Bernoulli( $C\xi$ );

- 1  $k \leftarrow 4.6/\zeta, \zeta \leftarrow \min\{\zeta, 0.644\}, i \leftarrow 1$ ;
- 2 **repeat**
- 3     **repeat**
- 4         draw  $B \leftarrow \mathcal{O}, G \leftarrow \text{Geometric}(\frac{C-1}{C})$ ;  
       //  $G$  is drawn according to geometric distribution with parameter  $\frac{C-1}{C}$
- 5          $i \leftarrow i - 1 + (1 - B)G$ ;
- 6     **until**  $i = 0$  or  $i \geq k$ ;
- 7     **if**  $i \geq k$  **then**
- 8         draw  $R \leftarrow \text{Bernoulli}((1 + \zeta/2)^{-i})$ ;
- 9          $C \leftarrow C(1 + \zeta/2), \zeta \leftarrow \zeta/2, k \leftarrow 2k$ ;
- 10 **until**  $i = 0$  or  $R = 0$ ;
- 11 **return**  $\mathbb{1}[i = 0]$ ;

---

**Proposition A.3** ([NP05, Proposition 14, (iv)]). *Given access to two coins  $\mathcal{O}_{\xi_1}$  and  $\mathcal{O}_{\xi_2}$ , and given as input  $\zeta > 0$ , with promise that  $\xi_1 - \xi_2 \geq \zeta$ , SubtractBF( $\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}, \zeta$ ) terminates with probability 1 and returns a draw of  $\mathcal{O}_{\xi_1 - \xi_2}$ .*

Recall the coins  $\mathcal{O}_{\theta_v}$  and  $\mathcal{O}_{v(x)}$  for  $x \in Q_v$  where  $v = \mu_v^\sigma$ . We further assume that the values in  $Q_v$  are enumerated in an arbitrary order as  $Q_v = \{x_1, x_2, \dots, x_{q_v}\}$ . To draw a sample from the distribution  $\mathcal{D}$  defined in (85), we construct the following Bernoulli factory:

- for each  $i \in [q_v]$ , let  $\mathcal{O}^i = \text{SubtractBF}(\mathcal{O}_{v(x_i)}, \mathcal{O}_{\theta_v}, \zeta)$ ;
- draw  $I \leftarrow \text{BernoulliRace}(\{\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^{q_v}\})$ , and return the  $I$ -th value  $x_I$  in  $Q_v$ .

The parameter  $\zeta > 0$  in above is as fixed in (7) and satisfies the promise (84) assuming Condition 3.8 (due to Corollary 4.4). Thus by Propositions A.1, A.2 and A.3, the above procedure terminates with probability 1 and returns an  $x_I$  distributed as  $\mathcal{D}$  defined as in (85). This proves Lemma 3.10.

**A.2. Efficiency of the Bernoulli Factory.** We now bound the efficiency of the Bernoulli factory constructed above. In this analysis, we need to explicitly bound the costs for realizations of the basic oracles  $\mathcal{O}_{v(x)}$  for  $x \in Q_v$  through the rejection sampling  $\mathcal{O}_{v(x)} = \mathbb{1}[\text{RejectionSampling}(\Phi, \sigma, \{v\}) = x]$ , whose complexity is measured in terms of both the computation cost and the query complexity for the evaluation oracle in Assumption 1.

Recall the simplification and decomposition of CSP defined in Section 3.2. Let  $\Phi^\sigma = (V^\sigma, \mathcal{Q}^\sigma, \mathcal{C}^\sigma)$  denote the simplification of  $\Phi$  under partial assignment  $\sigma \in \mathcal{Q}^*$ , and  $H^\sigma = H_{\Phi^\sigma} = (V^\sigma, \mathcal{C}^\sigma)$  its hypergraph representation. Recall that for each  $v \in V^\sigma$ ,  $H_v^\sigma = (V_v^\sigma, \mathcal{C}_v^\sigma)$  denotes the connected component in  $H^\sigma$  that contains the vertex/variable  $v$ . Let  $\Phi_v^\sigma$  be its corresponding formula.

We show the following theorem for upper bound on the complexity of the Bernoulli factory.

**Theorem A.4.** *Assuming Condition 3.8 for the input  $(\Phi, \sigma, v)$ , the Bernoulli factory algorithm constructed in Appendix A.1 costs in expectation:*

- $O\left(q^2 k^2 \Delta^6 (|\mathcal{C}_v^\sigma| + 1)(1 - \epsilon \alpha q)^{-|\mathcal{C}_v^\sigma|}\right)$  queries to the evaluation oracle in Assumption 1;
- $O\left(q^3 k^3 \Delta^6 (|\mathcal{C}_v^\sigma| + 1)(1 - \epsilon \alpha q)^{-|\mathcal{C}_v^\sigma|}\right)$  in computation.

To prove Theorem A.4, we first bound the cost for realizing the basic oracles  $\mathcal{O}_{v(x)}$  for  $x \in Q_v$ .

**Lemma A.5.** *Assume Condition 3.8 for the input  $(\Phi, \sigma, v)$ . It takes at most  $\Delta(|\mathcal{C}_v^\sigma| + 1)$  queries to the evaluation oracle in Assumption 1 and  $O(k\Delta(|\mathcal{C}_v^\sigma| + 1))$  computation cost for preprocessing the oracles  $\mathcal{O}_{v(x)}$  for all  $x \in Q_v$ . And upon each query,  $\mathcal{O}_{v(x)}$  returns using at most  $|\mathcal{C}_v^\sigma|(1 - \epsilon \alpha q)^{-|\mathcal{C}_v^\sigma|}$  queries to the evaluation oracle in expectation and  $O\left((qk(|\mathcal{C}_v^\sigma| + 1))(1 - \epsilon \alpha q)^{-|\mathcal{C}_v^\sigma|}\right)$  computation cost in expectation.*

*Proof.* The oracle  $\mathcal{O}_{v(x)}$  is computed as  $\mathcal{O}_{v(x)} = \mathbb{1} [\text{RejectionSampling}(\Phi, \sigma, \{v\}) = x]$ .

First, observe that  $\text{RejectionSampling}(\Phi, \sigma, \{v\})$  is equivalent to  $\text{RejectionSampling}(\Phi_v^\sigma, \sigma_{V \setminus \Lambda(\sigma)}, \{v\})$ . By using a depth-first search in  $H^\sigma$ , the connected component  $\Phi_v^\sigma = (V_v^\sigma, \mathcal{Q}_v^\sigma, \mathcal{C}_v^\sigma)$  can be constructed using at most  $\Delta(|\mathcal{C}_v^\sigma| + 1)$  queries to the evaluation oracle and  $O(\Delta|\mathcal{C}_v^\sigma| + \Delta|V_v^\sigma|) = O(k\Delta(|\mathcal{C}_v^\sigma| + 1))$  computation cost, because  $|V_v^\sigma| \leq k|\mathcal{C}_v^\sigma| + 1$ . This is the preprocessing cost.

Then a query to oracle  $\mathcal{O}_{v(x)}$  is reduced to a calling to  $\text{RejectionSampling}(\Phi_v^\sigma, \sigma_{V \setminus \Lambda(\sigma)}, \{v\})$  using Algorithm 2. In fact, Line 1 can be skipped and  $K = 1$  in Line 2 since the component  $\Phi_v^\sigma$  containing  $v$  has been explicitly constructed in the preprocessing. It is well known that the expected number of trials (the **repeat** loop in Line 5) taken by the rejection sampling until success is given by  $\mathbb{P}_{\Phi_v^\sigma}[\Omega_{\Phi_v^\sigma}]^{-1}$ , where  $\Omega_{\Phi_v^\sigma}$  is the set of satisfying assignments of  $\Phi_v^\sigma$  and hence  $\mathbb{P}_{\Phi_v^\sigma}[\Omega_{\Phi_v^\sigma}]$  gives the probability that a uniform random assignment is satisfying for  $\Phi_v^\sigma$ . By Theorem 4.1, assuming Condition 3.8,

$$\mathbb{P}_{\Phi_v^\sigma}[\Omega_{\Phi_v^\sigma}] \geq (1 - e\alpha q)^{|\mathcal{C}_v^\sigma|}.$$

The rejection sampling in  $\text{RejectionSampling}(\Phi_v^\sigma, \sigma_{V \setminus \Lambda(\sigma)}, \{v\})$  takes  $(1 - e\alpha q)^{-|\mathcal{C}_v^\sigma|}$  trials in expectation. And within each trial, it is easy to verify that it uses at most  $|\mathcal{C}_v^\sigma|$  queries to the evaluation oracle and  $O(k|\mathcal{C}_v^\sigma| + q|V_v^\sigma|) = O(qk(|\mathcal{C}_v^\sigma| + 1))$  computation cost. This proves the lemma.  $\square$

We then state known results for the efficiency of Bernoulli factories.

**Proposition A.6** ([DHKN17, Theorem 3.3]). *Given access to a list of coins  $L = \{\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^q\}$ , where for each  $i \in [q]$ , the probability of heads for  $\mathcal{O}^i$  is  $\xi_i$ , the expected number of queries to  $\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^q$  for executing  $\text{BernoulliRace}(L)$  is at most  $q/(\sum_{i=1}^q \xi_i)$ .*

**Proposition A.7** ([Hub16, Theorem 1]). *Given access to a coin  $\mathcal{O}_\xi$ , given as input  $C > 1, \zeta > 0$ , with the promise  $C\xi \leq 1 - \zeta$ , the expected number of queries to  $\mathcal{O}_\xi$  made in  $\text{LinearBF}(\mathcal{O}, C, \zeta)$  is at most  $9.5C/\zeta$ .*

By Proposition A.7, we have the following complexity bound for the Bernoulli factory for subtraction.

**Corollary A.8.** *Given access to two coins  $\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}$ , and given as input  $\zeta > 0$ , with the promise  $\xi_1 - \xi_2 \geq \zeta$ , the expected number of queries to  $\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}$  for executing  $\text{SubtractBF}(\mathcal{O}_{\xi_1}, \mathcal{O}_{\xi_2}, \zeta)$  is at most  $\frac{39\zeta^{-1}}{1 - (\xi_1 - \xi_2)}$ .*

*Proof of Theorem A.4.* Recall that our Bernoulli factory algorithm is  $\text{BernoulliRace}(\{\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^{q_v}\})$  where  $\mathcal{O}^i = \text{SubtractBF}(\mathcal{O}_{v(x_i)}, \mathcal{O}_{\theta_v}, \zeta)$  for the  $i$ -th value  $x_i \in Q_v$  and  $v = \mu_v^\sigma$ . By Proposition A.6 and Corollary A.8, the total number of queries to the basic oracles  $\mathcal{O}_{v(x)}$  for  $x \in Q_v$  is bounded by:

$$\frac{q_v}{\sum_{x \in Q_v} (\mu_v^\sigma(x) - \theta_v)} \cdot \left( \max_{x \in Q_v} \frac{39\zeta^{-1}}{1 - (\mu_v^\sigma(x) - \theta_v)} \right) \leq \frac{39}{\zeta^2(1 - 2\eta - \zeta)} = O(q^2 k^2 \Delta^6),$$

where the inequality is due to  $\mu_v^\sigma(x) \leq \theta_v + 2\eta + \zeta$  by Corollary 4.4 assuming Condition 3.8. The theorem then follows by applying Lemma A.5 and observing that the preprocessing costs are paid only once in the beginning.  $\square$

## APPENDIX B. BASIC PROPERTIES OF VARIABLE/CONSTRAINT ATTRIBUTES ALONG Path

In this section, we prove the technical lemma (Lemma 6.14) regarding the attributes of various variable/constraint sets  $V_{\star}^\sigma, \mathcal{C}_{\text{frozen}}^\sigma, \mathcal{C}_{\star\text{-con}}^\sigma, \mathcal{C}_{\star\text{-frozen}}^\sigma, \mathcal{C}_v^\sigma$  along Path, where  $\mathcal{C}_{\text{frozen}}^\sigma$  is defined in Definition 3.1,  $\mathcal{C}_v^\sigma$  in Section 3.2,  $\mathcal{C}_{\star\text{-con}}^\sigma$  in Definition 3.6, and  $V_{\star}^\sigma, \mathcal{C}_{\star\text{-frozen}}^\sigma$  in Definition 6.13.

Recall in Definition 3.6: for any  $\sigma \in \mathcal{Q}^*$ ,  $H_{\text{fix}}^\sigma$  denotes the sub-hypergraph of  $H^\sigma$  induced by  $V^\sigma \cap V_{\text{fix}}^\sigma$ . Recall in Section 3.2 that for each  $v \in V^\sigma$ ,  $H_v^\sigma = (V_v^\sigma, \mathcal{C}_v^\sigma)$  denotes the connected component in  $H^\sigma$  that contains the vertex/variable  $v$ . For each  $c \in \mathcal{C}$ , we denote the simplified constraint of  $c$  under  $\sigma$  as  $c^\sigma$ .

Note that Lemma 6.14 consists of three parts: monotonicity property, upper bound on  $|\mathcal{C}_v^{\sigma_\ell}|, |\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}|$ , and upper bound on length of  $\text{Path}(\sigma)$ . We prove Lemma 6.14 by showing these three parts in order.

**B.1. Proof of the monotonicity property.** The following lemma will be used in the proof of the monotonicity property in Lemma 6.14.

**Lemma B.1.** *Given  $\sigma \in \mathcal{Q}^*$  with  $\text{NextVar}(\sigma) = u \neq \perp$ , it holds that  $u \notin V_{\text{fix}}^\sigma$ .*

*Proof.* By  $u = \text{NextVar}(\sigma) \neq \perp$  and the definition of  $\text{NextVar}(\sigma)$  in Definition 3.6, we have  $u \in V_{\star\text{-inf}}^\sigma$ . Combining with the definition of  $V_{\star\text{-inf}}^\sigma$ , we have  $u \notin V_\star^\sigma$ . Combining with  $V_\star^\sigma \subseteq V^\sigma \cap V_{\text{fix}}^\sigma$ , we have  $u \notin V_{\text{fix}}^\sigma$ .  $\square$

The next lemma states a basic monotonicity property when extending some partial assignment  $\sigma$  on  $\text{NextVar}(\sigma)$ .

**Lemma B.2.** *Given  $\sigma \in \mathcal{Q}^*$  with  $\text{NextVar}(\sigma) = u \neq \perp$  and  $a \in \mathcal{Q}_u \cup \{\star\}$ , let  $\tau = \sigma_{u \leftarrow a}$ . it holds that*

$$V_\star^\sigma \subseteq V_\star^\tau, \quad \mathcal{C}_\mathcal{P}^\sigma \subseteq \mathcal{C}_\mathcal{P}^\tau,$$

where  $\mathcal{P}$  can be any attribute  $\mathcal{P} \in \{\text{frozen}, \star\text{-con}, \star\text{-frozen}\}$ .

*Proof.* At first, we prove  $V_\star^\sigma \subseteq V_\star^\tau$ . By  $\text{NextVar}(\sigma) = u \neq \perp$  and Lemma B.1, we have  $u \notin V_{\text{fix}}^\sigma$ . Combining with the definition of  $V_{\text{fix}}^\sigma$  in Definition 3.1, we have  $\sigma(u) \neq \star$ . Therefore it is straightforward by Definition 6.13 that  $V_\star^\sigma \subseteq V_\star^\tau$ .

Now we prove  $\mathcal{C}_{\text{frozen}}^\sigma \subseteq \mathcal{C}_{\text{frozen}}^\tau$ . For each  $c \in \mathcal{C}_{\text{frozen}}^\sigma$ , we have  $\text{vbl}(c) \subseteq V_{\text{fix}}^\sigma$ . Combining with  $u \notin V_{\text{fix}}^\sigma$ , we have  $u \notin \text{vbl}(c)$ , which says  $\tau_{\text{vbl}(c)} = \sigma_{\text{vbl}(c)}$ , hence  $c \in \mathcal{C}_{\text{frozen}}^\tau$  by the consistency assumption of frozen oracle in Assumption 2. In summary, we have  $\mathcal{C}_{\text{frozen}}^\sigma \subseteq \mathcal{C}_{\text{frozen}}^\tau$ .

In the next, we prove  $\mathcal{C}_{\star\text{-con}}^\sigma \subseteq \mathcal{C}_{\star\text{-con}}^\tau$ . For each  $c \in \mathcal{C}_{\star\text{-con}}^\sigma$ , by Definition 3.6, there exists some vertex  $v \in \text{vbl}(c) \cap V_{\star\text{-con}}^\sigma$ . By  $v \in V_{\star\text{-con}}^\sigma$ , we have there exists some variable  $v'$  such that  $\sigma(v') = \star$  and  $v$  and  $v'$  are connected in  $H_{\text{fix}}^\sigma$ . We claim that  $H_{\text{fix}}^\sigma$  is a sub-hypergraph of  $H_{\text{fix}}^\tau$ . Then we have  $v$  and  $v'$  are connected in  $H_{\text{fix}}^\tau$ . In addition, recall  $u \notin V_{\text{fix}}^\sigma$ . Combining with  $v' \in V_{\text{fix}}^\sigma$ , we have  $u \neq v'$  and then  $\tau(v') = \sigma(v') = \star$ . Combining with  $v$  and  $v'$  are connected in  $H_{\text{fix}}^\tau$ , we have  $v \in V_{\star\text{-con}}^\tau$ . Combining with  $v \in \text{vbl}(c)$ , we have  $c \in \mathcal{C}_{\star\text{-con}}^\tau$ . In summary, we have  $\mathcal{C}_{\star\text{-con}}^\sigma \subseteq \mathcal{C}_{\star\text{-con}}^\tau$ .

Now we prove the claim that  $H_{\text{fix}}^\sigma$  is a sub-hypergraph of  $H_{\text{fix}}^\tau$ . At first, we show

$$(86) \quad V^\sigma \cap V_{\text{fix}}^\sigma \subseteq V^\tau \cap V_{\text{fix}}^\tau.$$

Obviously,  $\Lambda^+(\sigma) \subseteq \Lambda^+(\tau)$ . Combining with  $\mathcal{C}_{\text{frozen}}^\sigma \subseteq \mathcal{C}_{\text{frozen}}^\tau$ , we have  $V_{\text{fix}}^\sigma \subseteq V_{\text{fix}}^\tau$ . In addition, we have

$$V^\sigma \cap V_{\text{fix}}^\sigma = (V^\sigma \setminus \{u\}) \cap V_{\text{fix}}^\sigma \subseteq V^\tau \cap V_{\text{fix}}^\sigma \subseteq V^\tau \cap V_{\text{fix}}^\tau,$$

where the first relation is by  $u \notin V_{\text{fix}}^\sigma$ , the second is by  $V^\sigma \setminus \{u\} \subseteq V^\tau$  and the last is by  $V_{\text{fix}}^\sigma \subseteq V_{\text{fix}}^\tau$ . Thus, we have (86) holds. In addition, let  $\mathcal{C}_{\text{fix}}^\sigma$  be the hyperedge set of  $H_{\text{fix}}^\sigma$  and  $\mathcal{C}_{\text{fix}}^\tau$  be the hyperedge set of  $H_{\text{fix}}^\tau$ . We can show that

$$\begin{aligned} & \mathcal{C}_{\text{fix}}^\sigma \\ (\text{definitions of } H_{\text{fix}}^\sigma \text{ and } \mathcal{C}_{\text{fix}}^\sigma) &= \{c^\sigma \mid (\text{vbl}(c^\sigma) \subseteq V^\sigma \cap V_{\text{fix}}^\sigma) \wedge (c \text{ is not satisfied by } \sigma)\} \\ & (\text{by } u \notin V_{\text{fix}}^\sigma) = \{c^\sigma \mid (\text{vbl}(c^\sigma) \subseteq V^\sigma \cap V_{\text{fix}}^\sigma) \wedge (u \notin \text{vbl}(c)) \wedge (c \text{ is not satisfied by } \sigma)\} \\ & (\text{by } c^\sigma = c^\tau \text{ if } u \notin \text{vbl}(c)) \subseteq \{c^\tau \mid (\text{vbl}(c^\tau) \subseteq V^\sigma \cap V_{\text{fix}}^\sigma) \wedge (u \notin \text{vbl}(c)) \wedge (c \text{ is not satisfied by } \tau)\} \\ & (\text{by (86)}) \subseteq \{c^\tau \mid (\text{vbl}(c^\tau) \subseteq V^\tau \cap V_{\text{fix}}^\tau) \wedge (u \notin \text{vbl}(c)) \wedge (c \text{ is not satisfied by } \tau)\} \\ & \subseteq \{c^\tau \mid (\text{vbl}(c^\tau) \subseteq V^\tau \cap V_{\text{fix}}^\tau) \wedge (c \text{ is not satisfied by } \tau)\} \\ (\text{definitions of } H_{\text{fix}}^\tau \text{ and } \mathcal{C}_{\text{fix}}^\tau) &= \mathcal{C}_{\text{fix}}^\tau \end{aligned}$$

Combining with (86), we have proven the claim that  $H_{\text{fix}}^\sigma$  is a sub-hypergraph of  $H_{\text{fix}}^\tau$ .

By  $\mathcal{C}_{\text{frozen}}^\sigma \subseteq \mathcal{C}_{\text{frozen}}^\tau$  and  $\mathcal{C}_{\star\text{-con}}^\sigma \subseteq \mathcal{C}_{\star\text{-con}}^\tau$ , we have

$$\mathcal{C}_{\star\text{-frozen}}^\sigma = \mathcal{C}_{\text{frozen}}^\sigma \cap \mathcal{C}_{\star\text{-con}}^\sigma \subseteq \mathcal{C}_{\text{frozen}}^\tau \cap \mathcal{C}_{\star\text{-con}}^\tau = \mathcal{C}_{\star\text{-frozen}}^\tau.$$

$\square$

By definition of  $\text{Path}(\cdot)$  and Lemma B.2, the monotonicity property in Lemma 6.14 is immediate by induction.

**B.2. Proof of the upper bound on  $|\mathcal{C}_v^{\sigma_\ell}|$  and  $|\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}|$ .** The following lemma will be used in the proof of the upper bound on  $|\mathcal{C}_v^{\sigma_\ell}|$  and  $|\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}|$  in Lemma 6.14.

**Lemma B.3.** *Let  $\sigma \in \mathcal{Q}^*$  be a partial assignment with exactly one variable  $v \in V$  having  $\sigma(v) = \star$  and  $\text{Path}(\sigma) = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ . For each  $c^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell}$  and each variable  $u \in \text{vbl}(c^{\sigma_\ell})$ , it holds that  $c \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$  and  $u \in V_{\text{fix}}^{\sigma_\ell}$ .*

*Proof.* For each simplified constraint  $c^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell}$ , by the definition of  $\mathcal{C}_v^{\sigma_\ell}$ , we have there exists a connected path  $c_1^{\sigma_\ell}, c_2^{\sigma_\ell}, \dots, c_t^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell}$  such that  $v \in \text{vbl}(c_1^{\sigma_\ell})$ ,  $c_t^{\sigma_\ell} = c^{\sigma_\ell}$  and  $c_i^{\sigma_\ell}$  intersects  $c_{i+1}^{\sigma_\ell}$  for each  $1 \leq i < t$ . Let  $\text{dist}(v, c^{\sigma_\ell})$ , or  $\text{dist}(v, c^{\sigma_\ell})$  for short, denote the length of the shortest connected path from  $v$  to  $c^{\sigma_\ell}$  in  $H_\ell^\sigma$ . We prove the lemma by induction on  $\text{dist}(v, c^{\sigma_\ell})$ .

For the base case when  $\text{dist}(v, c^{\sigma_\ell}) = 1$ , we have  $v \in \text{vbl}(c^{\sigma_\ell})$ . In addition, by  $\sigma_\ell(v) = \sigma(v) = \star$ , we have  $v \in V^{\sigma_\ell} \cap V_{\text{fix}}^{\sigma_\ell}$ . Thus, we have  $v \in V_{\star\text{-con}}^{\sigma_\ell}$ . Combining with  $v \in \text{vbl}(c^{\sigma_\ell})$ , we have  $c \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$  by Definition 3.6. In addition, we have  $u \in V_{\text{fix}}^{\sigma_\ell}$  for each  $u \in \text{vbl}(c^{\sigma_\ell})$ . Because otherwise,  $u \notin V_{\text{fix}}^{\sigma_\ell}$ . We have  $u \in \text{vbl}(c^{\sigma_\ell}) \setminus V_{\text{fix}}^{\sigma_\ell} \subseteq V^{\sigma_\ell} \setminus V_{\star\text{-con}}^{\sigma_\ell}$  by  $V_{\star\text{-con}}^{\sigma_\ell} \subseteq V_{\text{fix}}^{\sigma_\ell}$ . In addition, by  $u, v \in \text{vbl}(c^{\sigma_\ell})$ ,  $v \in V_{\star\text{-con}}^{\sigma_\ell}$ , and  $c^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell} \subseteq \mathcal{C}^{\sigma_\ell}$ , we have  $u \in V_{\star\text{-inf}}^{\sigma_\ell}$ . Therefore, we have  $V_{\star\text{-inf}}^{\sigma_\ell} \neq \emptyset$  and  $\text{NextVar}(\sigma_\ell) \neq \perp$ . Thus, by Definition 6.10, there must be another partial assignment  $\sigma_{\ell+1}$  generated from  $\sigma_\ell$ , which is contradictory with  $\text{Path}(\sigma) = (\sigma_0, \dots, \sigma_\ell)$ .

For the induction step, we assume  $\text{dist}(v, c^{\sigma_\ell}) = t > 1$ . Let  $c_1^{\sigma_\ell}, c_2^{\sigma_\ell}, \dots, c_t^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell}$  be a connected path in  $H_v^{\sigma_\ell}$  with  $v \in \text{vbl}(c_1^{\sigma_\ell})$ ,  $c_t^{\sigma_\ell} = c^{\sigma_\ell}$  and  $c_i^{\sigma_\ell}$  intersects  $c_{i+1}^{\sigma_\ell}$  for each  $1 \leq i < t$ . By the induction hypothesis and  $c_i^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell}$ , we have  $c_i \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$  for each  $i < t$ . Choose some  $w \in \text{vbl}(c_{t-1}^{\sigma_\ell}) \cap \text{vbl}(c_t^{\sigma_\ell})$ . By the induction hypothesis, we have  $w \in V_{\text{fix}}^{\sigma_\ell}$ . Combining with  $w \in V^{\sigma_\ell}$ , we have  $w \in V_{\text{fix}}^{\sigma_\ell} \cap V^{\sigma_\ell}$ . Recall that  $\sigma_\ell(v) = \star$ . Combining with  $w \in V_{\text{fix}}^{\sigma_\ell} \cap V^{\sigma_\ell}$  and that  $w \in \text{vbl}(c_{t-1}^{\sigma_\ell})$  is connected to  $v \in \text{vbl}(c_1^{\sigma_\ell})$  by the path  $c_1^{\sigma_\ell}, c_2^{\sigma_\ell}, \dots, c_{t-1}^{\sigma_\ell}$  in  $H_v^{\sigma_\ell}$ , we have  $w \in V_{\star\text{-con}}^{\sigma_\ell}$ . Combining with  $w \in \text{vbl}(c_t^{\sigma_\ell})$ , we have  $c_t \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$ . In addition, we have  $u \in V_{\text{fix}}^{\sigma_\ell}$  for each  $u \in \text{vbl}(c^{\sigma_\ell})$ . Because otherwise,  $u \notin V_{\text{fix}}^{\sigma_\ell}$ . We have  $u \in \text{vbl}(c^{\sigma_\ell}) \setminus V_{\text{fix}}^{\sigma_\ell} \subseteq V^{\sigma_\ell} \setminus V_{\star\text{-con}}^{\sigma_\ell}$  by  $V_{\star\text{-con}}^{\sigma_\ell} \subseteq V_{\text{fix}}^{\sigma_\ell}$ . In addition, by  $u, w \in \text{vbl}(c^{\sigma_\ell})$ ,  $w \in V_{\star\text{-con}}^{\sigma_\ell}$ , and  $c^{\sigma_\ell} \in \mathcal{C}_v^{\sigma_\ell} \subseteq \mathcal{C}^{\sigma_\ell}$ , we have  $u \in V_{\star\text{-inf}}^{\sigma_\ell}$ . Therefore, similar to the base case one can also reach a contradiction. This completes the induction step and the proof of the lemma.  $\square$

Because  $|\mathcal{C}_v^{\sigma_\ell}| \leq |\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}|$  is immediate by Lemma B.3, it is sufficient to show that  $|\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}| \leq \Delta \cdot |\mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}| + \Delta \cdot |V_\star^{\sigma_\ell}|$ . We show this by proving that for each  $c \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$ , either there exists some  $u \in \text{vbl}(c)$  such that  $u \in V_\star^{\sigma_\ell}$ , or there exists some  $c' \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}$  such that  $\text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset$ .

For each  $c \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$ , by Definition 3.6, we have there exists some  $u \in V_{\star\text{-con}}^{\sigma_\ell} \cap \text{vbl}(c^{\sigma_\ell})$ . By  $u \in V_{\star\text{-con}}^{\sigma_\ell}$ , we have  $u \in V^{\sigma_\ell} \cap V_{\text{fix}}^{\sigma_\ell}$ . By  $u \in V^{\sigma_\ell}$ , we have  $u \notin \Lambda(\sigma_\ell)$ . Combining with  $u \in V_{\text{fix}}^{\sigma_\ell}$ , we have either  $\sigma_\ell(u) = \star$  or  $u \in c'$  for some  $c' \in \mathcal{C}_{\text{frozen}}^{\sigma_\ell}$ . If  $\sigma_\ell(u) = \star$ , we have  $u \in V_\star^{\sigma_\ell}$ . Otherwise,  $u \in c'$  for some  $c' \in \mathcal{C}_{\text{frozen}}^{\sigma_\ell}$ . In addition, we also have  $c' \in \mathcal{C}_{\star\text{-con}}^{\sigma_\ell}$  by  $u \in V_{\star\text{-con}}^{\sigma_\ell}$  and  $u \in \text{vbl}(c')$ . Combining with  $c' \in \mathcal{C}_{\text{frozen}}^{\sigma_\ell}$ , we have  $c' \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}$ . This completes the proof of the upper bound on  $|\mathcal{C}_v^{\sigma_\ell}|$  and  $|\mathcal{C}_{\star\text{-con}}^{\sigma_\ell}|$  in Lemma 6.14.

**B.3. Proof of the upper bound on length of  $\text{Path}(\sigma)$ .** Fix any  $0 \leq i \leq \ell$ . We claim that for each  $0 \leq j < i$ ,

- (1) either there exist some  $c_j, c'_j$  such that  $\text{NextVar}(\sigma_j) \in \text{vbl}(c_j)$ ,  $c'_j \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_i}$ , and  $\text{vbl}(c_j) \cap \text{vbl}(c'_j) \neq \emptyset$ ;
- (2) or there exist some  $c_j, u_j$  such that  $\text{NextVar}(\sigma_j) \in \text{vbl}(c_j)$  and  $u_j \in V_\star^{\sigma_i}$ .

Therefore, for each  $0 \leq j < i$ ,  $\text{NextVar}(\sigma_j)$  is in a constraint  $c$  where either  $\text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset$  for some  $c' \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_i}$ , or  $u \in \text{vbl}(c)$  for some  $u \in V_\star^{\sigma_i}$ . By Lemma B.2 and induction, we have  $\mathcal{C}_{\star\text{-frozen}}^{\sigma_i} \subseteq \mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell}$  and  $V_\star^{\sigma_i} \subseteq V_\star^{\sigma_\ell}$  for each  $0 \leq i \leq \ell$ . Combining with  $|\text{vbl}(c)| \leq k$ , we have

$$\begin{aligned} \ell &\leq k \cdot \left| \{c \in \mathcal{C} : \text{vbl}(c) \cap \text{vbl}(c') \neq \emptyset \text{ for some } c' \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_\ell} \text{ or } u \in \text{vbl}(c) \text{ for some } u \in V_\star^{\sigma_\ell}\} \right| \\ &\leq k\Delta \cdot \left( |\mathcal{C}_{\star\text{-frozen}}^{\sigma_i}| + |V_\star^{\sigma_i}| \right), \end{aligned}$$

which proves the upper bound on length of  $\text{Path}(\sigma)$  in Lemma 6.14.

Now we prove the claim. Note that by  $\text{Path}(\sigma) = (\sigma_0, \dots, \sigma_\ell)$ ,  $0 \leq i \leq \ell$  and Definition 6.10, we have  $\text{NextVar}(\sigma_j) \neq \perp$  for each  $0 \leq j < i$ . Assume that  $\text{NextVar}(\sigma_j) = u_j$ . By Definition 3.6, we have  $u_j \in V_{\star\text{-inf}}^{\sigma_j} \neq \emptyset$ . Combining with the definition of  $V_{\star\text{-inf}}^{\sigma_j}$ , we have there exists some  $c_j \in \mathcal{C}^{\sigma_j}$ ,  $w_j \in V_{\star\text{-con}}^{\sigma_j}$  such that  $u_j, w_j \in \text{vbl}(c_j)$ . By  $w_j \in V_{\star\text{-con}}^{\sigma_j}$ , we have  $w_j \in V^{\sigma_j} \cap V_{\text{fix}}^{\sigma_j}$ . By  $w_j \in V^{\sigma_j}$ , we have  $w_j \notin \Lambda(\sigma_j)$ . Combining with  $w_j \in V_{\text{fix}}^{\sigma_j}$ , we have either  $\sigma_j(w_j) = \star$  or  $w_j \in \widehat{c}_j$  for some  $\widehat{c}_j \in \mathcal{C}_{\text{frozen}}^{\sigma_j}$ . If  $\sigma_j(w_j) = \star$ , we have  $w_j \in V_{\star}^{\sigma_j} \subseteq V_{\star}^{\sigma_i}$  and  $c_j, w_j$  satisfies Item 2. Otherwise,  $w_j \in \text{vbl}(\widehat{c}_j)$  for some  $\widehat{c}_j \in \mathcal{C}_{\text{frozen}}^{\sigma_j}$ . In addition, by  $w_j \in V_{\star\text{-con}}^{\sigma_j}$  and  $w_j \in \text{vbl}(\widehat{c}_j)$ , we have  $\widehat{c}_j \in \mathcal{C}_{\star\text{-con}}^{\sigma_j}$ . Combining with  $\widehat{c}_j \in \mathcal{C}_{\text{frozen}}^{\sigma_j}$ , we have  $\widehat{c}_j \in \mathcal{C}_{\star\text{-frozen}}^{\sigma_j}$ . By  $w_j \in \text{vbl}(c_j)$  and  $w_j \in \text{vbl}(\widehat{c}_j)$ , we have  $\text{vbl}(c_j) \cap \text{vbl}(\widehat{c}_j) \neq \emptyset$  and  $c_j, \widehat{c}_j$  satisfies Item 1. This justifies the claim and finishes the proof of Lemma 6.14.